On intersection forms of definite 4-manifolds bounded by a rational homology 3-sphere

Dong Heon Choe (joint work with Kyungbae Park)

Seoul National University

East Asian Conference on Gauge Theory and Related Topics

September 12, 2018

Definitions

- Let Y be a oriented closed 3-manifold. Y is a rational homology 3-sphere($\mathbb{Q}HS^3$) if $H_*(Y, \mathbb{Q}) \cong H_*(S^3, \mathbb{Q})$.
  e.g. lens space $L(p, q)$, Brieskorn sphere $\Sigma(p, q, r)$.

Definitions

- Let Y be a oriented closed 3-manifold. Y is a rational homology 3-sphere($\mathbb{Q}HS^3$) if $H_*(Y,\mathbb{Q}) \cong H_*(S^3,\mathbb{Q})$.
  e.g. lens space $L(p,q)$, Brieskorn sphere $\Sigma(p,q,r)$.
- Let X be an oriented connected compact 4-manifold (possibly with boundary). The *intersection form* of X is an integral symmetric bilinear form

$$Q_X : H_2(X;\mathbb{Z})/Tors \times H_2(X;\mathbb{Z})/Tors \to \mathbb{Z}$$

  given by Lefschetz duality and cup product.
  e.g. S^4, $S^2 \times S^2$, $\mathbb{CP}^2$.

Main question

Question

For a given rational homology 3-sphere Y, which nondegenerate definite bilinear forms are realized as the intersection form of a smooth 4-manifold bounded by Y?

Related results

- (S. Donaldson, 1982) Any simply connected closed smooth definite 4-manifold has diagonalizable intersection form.

Related results

- (S. Donaldson, 1982) Any simply connected closed smooth definite 4-manifold has diagonalizable intersection form.
- (K. Frøyshov, Ph.D thesis) A negative definite simply connected smooth 4-manifold bounded by the Poincaré homology sphere has intersection form which is diagonalizable or equivalent to $-E_8 \oplus \langle -1 \rangle^m$ for some m.

Related results

- (S. Donaldson, 1982) Any simply connected closed smooth definite 4-manifold has diagonalizable intersection form.
- (K. Frøyshov, Ph.D thesis) A negative definite simply connected smooth 4-manifold bounded by the Poincaré homology sphere has intersection form which is diagonalizable or equivalent to $-E_8 \oplus \langle -1 \rangle^m$ for some m.
- (B. Owens and S. Strle, 2011) determined which Dehn surgery 3-manifolds along the torus knots bound a negative definite smooth 4-manifold.

Related results

- (S. Donaldson, 1982) Any simply connected closed smooth definite 4-manifold has diagonalizable intersection form.
- (K. Frøyshov, Ph.D thesis) A negative definite simply connected smooth 4-manifold bounded by the Poincaré homology sphere has intersection form which is diagonalizable or equivalent to $-E_8 \oplus \langle -1 \rangle^m$ for some m.
- (B. Owens and S. Strle, 2011) determined which Dehn surgery 3-manifolds along the torus knots bound a negative definite smooth 4-manifold.
- (C. Scaduto, 2018) extended the results of Frøyshov for the 3-manifolds which are obtained by Dehn surgery along a knot with 4-ball genus 1 or 2.

Integral lattices

An integral lattice $\Lambda := (\mathbb{Z}^n, Q)$ is a free abelian group with nondegenerate integral symmetric bilinear form Q on $\mathbb{Z}^n$. We say

- Λ is even if $Q(v, v)$ is even for any $v \in \mathbb{Z}^n$.
- Λ is odd if $Q(v, v)$ is odd for some $v \in \mathbb{Z}^n$.
- Λ is definite if $|sign(Q)| = n$
- Λ is unimodular if $|\det(Q)| = 1$

Definitions

- We say that an integral lattice Λ is smoothly (topologically) bounded by a 3-manifold Y if there is a smooth (topological) 4-manifold X with boundary Y and $(\mathbb{Z}^{b_2(X)}, Q_X) \cong \Lambda$.

Definitions

- We say that an integral lattice Λ is smoothly (topologically) bounded by a 3-manifold Y if there is a smooth (topological) 4-manifold X with boundary Y and $(\mathbb{Z}^{b_2(X)}, Q_X) \cong \Lambda$.
- Two negative definite lattices Λ_1 and Λ_2 are stable equivalent if $\Lambda_1 \oplus \langle -1 \rangle^n \cong \Lambda_2 \oplus \langle -1 \rangle^m$ for some non-negative integers n and m.

Definitions

- We say that an integral lattice Λ is smoothly (topologically) bounded by a 3-manifold Y if there is a smooth (topological) 4-manifold X with boundary Y and $(\mathbb{Z}^{b_2(X)}, Q_X) \cong \Lambda$.
- Two negative definite lattices Λ_1 and Λ_2 are stable equivalent if $\Lambda_1 \oplus \langle -1 \rangle^n \cong \Lambda_2 \oplus \langle -1 \rangle^m$ for some non-negative integers n and m.
- Let $\mathcal{I}(Y)(\mathcal{I}^{TOP}(Y))$ be the set of all negative definite lattices that can be smoothly (topologically) bounded by Y, up to stable-equivalence.

Definitions

- We say that an integral lattice Λ is smoothly (topologically) bounded by a 3-manifold Y if there is a smooth (topological) 4-manifold X with boundary Y and $(\mathbb{Z}^{b_2(X)}, Q_X) \cong \Lambda$.
- Two negative definite lattices Λ_1 and Λ_2 are stable equivalent if $\Lambda_1 \oplus \langle -1 \rangle^n \cong \Lambda_2 \oplus \langle -1 \rangle^m$ for some non-negative integers n and m.
- Let $\mathcal{I}(Y)(\mathcal{I}^{TOP}(Y))$ be the set of all negative definite lattices that can be smoothly (topologically) bounded by Y, up to stable-equivalence.
- $\mathcal{I}_s(Y)$ is same as $\mathcal{I}(Y)$ except additional simply connected condition on the bounding 4-manifolds

Examples

- $\mathcal{I}^{TOP}(Y) = \{\text{all negative definite unimodular lattices}\}/\sim$ for an $\mathbb{Z}HS^3$ Y (Freedman)

Examples

- $\mathcal{I}^{TOP}(Y) = \{\text{all negative definite unimodular lattices}\}/\sim$ for an $\mathbb{Z}HS^3$ Y (Freedman)
- $|\mathcal{I}^{TOP}(Y)| = \infty$ for a $\mathbb{Q}HS^3$ Y (Boyer and Edmond)

Examples

- $\mathcal{I}^{TOP}(Y) = \{\text{all negative definite unimodular lattices}\}/\sim$ for an $\mathbb{Z}HS^3$ Y (Freedman)
- $|\mathcal{I}^{TOP}(Y)| = \infty$ for a $\mathbb{Q}HS^3$ Y (Boyer and Edmond)
- $\mathcal{I}(S^3) = \{[\langle -1 \rangle]\}$ (Donaldson)

Examples

- $\mathcal{I}^{TOP}(Y) = \{\text{all negative definite unimodular lattices}\}/\sim$ for an $\mathbb{Z}HS^3$ Y (Freedman)
- $|\mathcal{I}^{TOP}(Y)| = \infty$ for a $\mathbb{Q}HS^3$ Y (Boyer and Edmond)
- $\mathcal{I}(S^3) = \{[\langle -1 \rangle]\}$ (Donaldson)
- $\mathcal{I}_s(\Sigma(2,3,5)\#(-\Sigma(2,3,5))) = \emptyset$ (Taubes)

Examples

- $\mathcal{I}^{TOP}(Y) = \{\text{all negative definite unimodular lattices}\}/\sim$ for an $\mathbb{Z}HS^3$ Y (Freedman)
- $|\mathcal{I}^{TOP}(Y)| = \infty$ for a $\mathbb{Q}HS^3$ Y (Boyer and Edmond)
- $\mathcal{I}(S^3) = \{[\langle -1\rangle]\}$ (Donaldson)
- $\mathcal{I}_s(\Sigma(2,3,5)\#(-\Sigma(2,3,5))) = \emptyset$ (Taubes)
- $\mathcal{I}_s(\Sigma(2,3,5)) = \{[\langle -1\rangle], [-E_8]\}$(Frøyshov and Scaduto)

Examples

- $\mathcal{I}^{TOP}(Y) = \{\text{all negative definite unimodular lattices}\}/\sim$ for an $\mathbb{Z}HS^3$ Y (Freedman)
- $|\mathcal{I}^{TOP}(Y)| = \infty$ for a $\mathbb{Q}HS^3$ Y (Boyer and Edmond)
- $\mathcal{I}(S^3) = \{[\langle -1 \rangle]\}$ (Donaldson)
- $\mathcal{I}_s(\Sigma(2,3,5)\#(-\Sigma(2,3,5))) = \emptyset$ (Taubes)
- $\mathcal{I}_s(\Sigma(2,3,5)) = \{[\langle -1 \rangle], [-E_8]\}$(Frøyshov and Scaduto)
- $2 \leq |\mathcal{I}(\Sigma(2,3,5))| \leq 15$(Frøyshov and Elkies)

Examples

- $\mathcal{I}^{TOP}(Y) = \{\text{all negative definite unimodular lattices}\}/\sim$ for an $\mathbb{Z}HS^3$ Y (Freedman)
- $|\mathcal{I}^{TOP}(Y)| = \infty$ for a $\mathbb{Q}HS^3$ Y (Boyer and Edmond)
- $\mathcal{I}(S^3) = \{[\langle -1 \rangle]\}$ (Donaldson)
- $\mathcal{I}_s(\Sigma(2,3,5)\#(-\Sigma(2,3,5))) = \emptyset$ (Taubes)
- $\mathcal{I}_s(\Sigma(2,3,5)) = \{[\langle -1 \rangle], [-E_8]\}$(Frøyshov and Scaduto)
- $2 \leq |\mathcal{I}(\Sigma(2,3,5))| \leq 15$(Frøyshov and Elkies)
- K. Sato and M. Taniguchi gave interesting examples.

Examples

- $\mathcal{I}^{TOP}(Y) = \{\text{all negative definite unimodular lattices}\}/\sim$ for an $\mathbb{Z}HS^3$ Y (Freedman)
- $|\mathcal{I}^{TOP}(Y)| = \infty$ for a $\mathbb{Q}HS^3$ Y (Boyer and Edmond)
- $\mathcal{I}(S^3) = \{[\langle -1 \rangle]\}$ (Donaldson)
- $\mathcal{I}_s(\Sigma(2,3,5)\#(-\Sigma(2,3,5))) = \emptyset$ (Taubes)
- $\mathcal{I}_s(\Sigma(2,3,5)) = \{[\langle -1 \rangle], [-E_8]\}$(Frøyshov and Scaduto)
- $2 \leq |\mathcal{I}(\Sigma(2,3,5))| \leq 15$(Frøyshov and Elkies)
- K. Sato and M. Taniguchi gave interesting examples.

Question

For a given rational homology 3-sphere Y, is $\mathcal{I}(Y)$ a finite set or not?

Main theorem

Theorem (K. Park-C, 17)

Let Y_1 and Y_2 be rational homology 3-spheres. Suppose that there is a negative definite cobordism from Y_1 to Y_2 and $|\mathcal{I}(Y_2)| < \infty$. Then $|\mathcal{I}(Y_1)| < \infty$

From the theorem, we can obtain finiteness results for more general rational homology 3-spheres.

Main theorem

If we assign Y_2 to S^3, then we obtain following corollary.

Corollary

If a rational homology 3-sphere Y bounds a positive definite smooth 4-manifold, then there are only finitely many negative definite lattices, up to stable-equivalence, which can be realized as the intersection form of a smooth 4-manifold bounded by Y. In other words, if $\mathcal{I}(-Y) \neq \emptyset$, then $|\mathcal{I}(Y)| < \infty$.

δ-invariant of lattices

We call $\xi \in \Lambda^*$ a characteristic covector if $\xi(w) \equiv Q(w, w) \pmod 2$ for any $w \in \Lambda$. We denote the set of characteristic covectors by $\mathrm{Char}(\Lambda)$.

Definition

Let Λ be a integral definite lattice.

$$\delta(\Lambda) := \max_{\xi \in \mathrm{Char}(\Lambda)} \left(\frac{\mathrm{rk}(\Lambda) - |\xi \cdot \xi|}{4} \right)$$

For example, if $\Lambda \cong \langle -1 \rangle^n$, then $\delta(\Lambda) = 0$. If Λ is even lattice, then $\delta(\Lambda) = \frac{1}{4}(\mathrm{rank}(\Lambda))$

δ-invariant of lattices

N. Elkies showed that the δ-invariant characterizes the standard definite lattices.

Theorem (N. Elkies, '95)

Let Λ be a negative definite unimodular lattice. Then $\delta(\Lambda) \geq 0$. Moreover $\delta(\Lambda) = 0$ if and only if $\Lambda \cong \langle -1 \rangle^n$ for some n.

Correction term invariant

Ozsváth and Szabó introduced the rational valued invariant called correction term invariant by using the TQFT properties of Heegaard Floer homology.

- It is denoted by $d(Y, \mathfrak{t}) \in \mathbb{Q}$ for spin^c $\mathbb{Q}$HS $(Y, \mathfrak{t})$.
- $d(Y_1 \# Y_2, \mathfrak{t}_1 \# \mathfrak{t}_2) = d(Y_1, \mathfrak{t}_1) + d(Y_2, \mathfrak{t}_2)$

Correction term invariant

The correction term also gives a constraint on the intersection form of a negative definite 4-manifold with a given boundary.

Theorem (Ozsváth-Szabó, '03)

If X is a negative definite smooth 4-manifold bounded by Y, then for each spin^c structure $\mathfrak{s}$ over X,

$$c_1(\mathfrak{s})^2 + b_2(X) \leq 4d(Y, \mathfrak{s}|_Y)$$

Correction term invariant

Note that $c_1(\mathfrak{s})$ is an integral lift of the second Stiefel-Whitney class. Hence, it is a characteristic covector of the intersection lattice of X.

Corollary

Suppose that a negative definite lattice Λ is bounded by a rational homology 3-sphere Y. Then

$$\delta(\Lambda) \leq \max_{\mathfrak{t} \in Spin^c(Y)} d(Y, \mathfrak{t})$$

Correction term invariant

Combining with Elkies's theorem, we obtain following immediate corollary. We denote the lattice induced from a 4-manifold X by $\Lambda_X := (\mathbb{Z}^{b_2(X)}, Q_X)$.

Corollary

Let Y be a integral homology 3-sphere. Suppose that there is a negative definite smooth 4-manifolds X with $\partial X \cong Y$. Then $d(Y) \geq 0$ Moreover, if $d(Y) = 0$, then Λ_X is diagonalizable.

Main theorem

To prove the main theorem, we consider a set of lattices defined purely algebraically in terms of the invariants of a given 3-manifolds.

Theorem

Let Γ_1 and Γ_2 be fixed negative definite lattices, and $C > 0$ and $D \in \mathbb{Z}$ be constants. Define $\mathcal{L}(\Gamma_1, \Gamma_2; C, D)$ to be the set of negative definite lattices Λ, up to the stable-equivalence, satisfying the following conditions:

- $\det(\Lambda) = D$,
- $\delta(\Lambda) \leq C$, *and*
- $\Gamma_1 \oplus \Lambda$ *embeds into* $\Gamma_2 \oplus \langle -1 \rangle^N$, $N = rk(\Gamma_1) + rk(\Lambda) - rk(\Gamma_2)$.

Then $\mathcal{L}(\Gamma_1, \Gamma_2; C, D)$ is finite.

Proof of the theorem

For simplicity, we show a special case of the theorem in which Γ_1 and Γ_2 are trivial lattice.

Proposition

Let $C > 0$ and $D \in \mathbb{Z}$ be constants. There are finitely many negative definite lattices Λ, up to the stable-equivalence, which satisfy the following conditions:

- $\det \Lambda = D$,
- $\delta(\Lambda) \leq C$, *and*
- Λ *is embedded into* $\langle -1 \rangle^{rk(\Lambda)}$ *with a prime index.*

Proof of the theorem

We assume that Λ has no square -1 vector and p is odd prime. From the third condition, we can write the basis vectors of Λ in term of the standard basis of $\langle -1 \rangle^{\mathrm{rk}(\Lambda)=n}$ as follows.

$$\mathcal{B} := \{pe, e_1 + s_1 e, \ldots, e_{n-1} + s_{n-1} e\}$$

where $e, e_1, \cdots, e_{n-1}$ be the standard basis of $\langle -1 \rangle^n$ and p is the prime index. we can also choose odd s_i such that $-p+1 < s_i < p-1$ for each i.

Proof of the theorem

Hence, the matrix representation of Λ is following.

$$Q = -\begin{pmatrix} p^2 & ps_1 & ps_2 & \dots & ps_{n-1} \\ ps_1 & 1+s_1^2 & s_1 s_2 & \dots & s_1 s_{n-1} \\ ps_2 & s_1 s_2 & 1+s_2^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & s_{n-2}s_{n-1} \\ ps_{n-1} & s_1 s_{n-1} & \dots & s_{n-2}s_{n-1} & 1+s_{n-1}^2 \end{pmatrix}.$$

Hence, a characteristic covector can be written as a vector

$$\xi = (k, k_1, \dots, k_{n-1}),$$

where k is an odd integer and k_i's are even integers, in terms of the dual basis of Q

Proof of the theorem

From the matrix Q^{-1}, we compute

$$|\xi \cdot \xi| = \frac{1}{p^2}(k^2 + \sum_{i=1}^{n-1}(ks_i - pk_i)^2).$$

Now, we use the following algebraic lemma to obtain a upper bound of $|\xi \cdot \xi|$.

Lemma

For an odd prime p and odd integers $s_1, s_2, \ldots, s_{n-1}$ in $[-p+1, p-1]$, there exists an odd integer k and even integers $k_1, k_2, \ldots, k_{n-1}$ such that

$$k^2 + \sum_{i=1}^{n-1}(ks_i - pk_i)^2 < \frac{n+2}{3}p^2.$$

Idea of proof : take average on $k \in K := \{-p+2, -p+4, \ldots, p-2\}$

Proof of the theorem

Hence we obtain

$$\min\{|\xi \cdot \xi| : \xi \text{ characteristic covector of } \Lambda\} \leq \frac{n+2}{3}.$$

Therefore, from

$$\delta(\Lambda) = \frac{1}{4}(n - \min_{\xi \in Char(\Lambda)} |\xi \cdot \xi|) \leq C,$$

we conclude that

$$\text{rk}(\Lambda) = n \leq 6C + 1.$$

It is known that there are only finitely many equivalence classes of lattices for the given rank and determinant.

Seifert 3-manifolds

A Seifert fibered rational homology 3-sphere can be represented by a Seifert form

$$(e_0; (a_1, b_1), \ldots, (a_k, b_k)),$$

where e_0, a_is are integers, b_is are positive integers and $\gcd(a_i, b_i) = 1$.

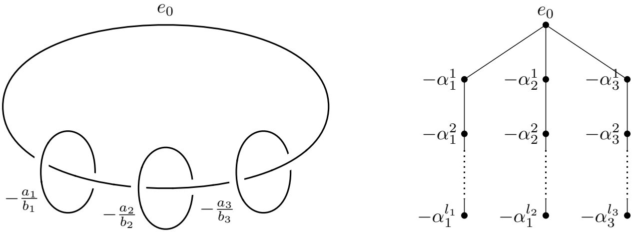

Figure: $M(e_0; (a_1, b_1), (a_2, b_2), (a_3, b_3))$.

Seifert 3-manifolds

Note that any Seifert fibered rational homology 3-sphere admits a canonical Seifert form,

$$(e_0; (a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k))$$

such that $a_i > b_i > 0$ for all $1 \leq i \leq k$. We refer to the form as the *normal form* of a Seifert fibered rational homology 3-sphere.

Proposition

Let Y be a Seifert fibered rational homology 3-sphere of the normal form

$$(e_0; (a_1, b_2), \ldots, (a_k, b_k)).$$

If $e_0 + k \leq 0$, then Y bounds both positive and negative definite smooth 4-manifolds, i.e., both $\mathcal{I}(Y)$ and $\mathcal{I}(-Y)$ are not empty.

Spherical 3-manifolds

Proposition

Any spherical 3-manifolds except T_1, O_1, I_1 and I_7 can bound both positive and negative definite smooth 4-manifolds. The manifolds T_1, O_1, I_1 and I_7 cannot bound a positive definite smooth 4-manifold.

Hence, to prove the finiteness for all spherical 3-manifolds, we need to find a negative definite cobordism from the exceptional cases to the known 3-manifold Y_0 with $|\mathcal{I}(Y_0)| < \infty$. Fortunately, by N. Elkies, it is known that $|\mathcal{I}(\Sigma(2,3,5))| < 15$. Note that the I_1-type manifold homeomorphic to $\Sigma(2,3,5)$.

Spherical 3-manifolds

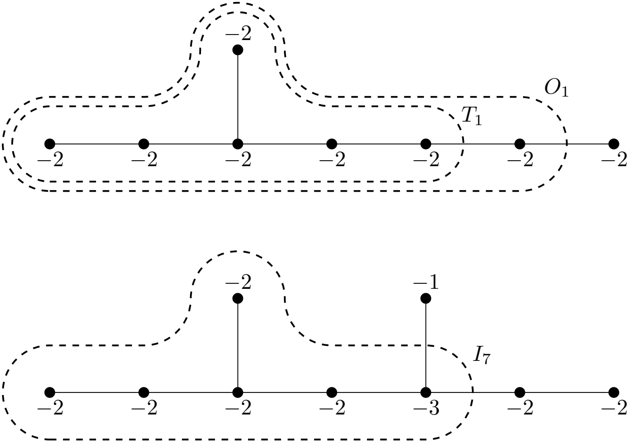

Figure: The embedding of the plumbed 4-manifold corresponding to the manifolds T_1, O_1 and I_7 into $-E_8$-manifold and $-E_8\#\overline{\mathbb{CP}^2}$.

Further Questions

By using topological obstruction, Donaldson obstruction and correction term invariants, one can define $\mathcal{L}(Y)$ as set of lattices satisfying all conditions for a $\mathbb{Q}HS^3$ Y.

- finiteness property for more general 3-manifolds
- $\mathcal{L}(Y) = \mathcal{I}(Y)$?
- $\mathcal{I}(\Sigma(2,3,5)) = ?$

Thank you for your attention!