Irreducible 3-manifolds that cannot be obtained by 0-surgery along a knot

Kyungbae Park (SNU)

joint with Matt Hedden (MSU), Min Hoon Kim (KIAS), and Tom Mark (UVA)

East Asian Conference on Gauge theory and Related topics

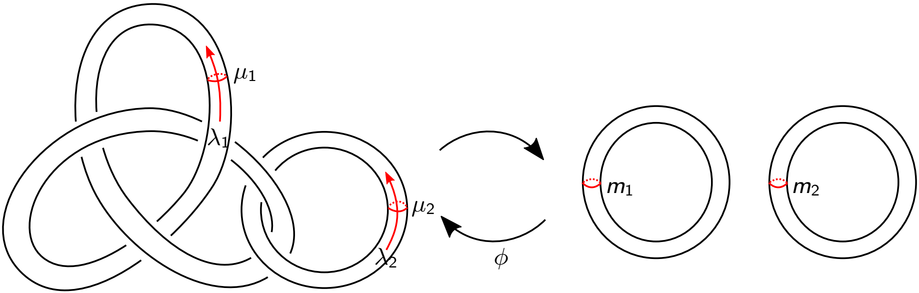
Dehn Surgery

Dehn surgery constructs closed, oriented 3-manifolds by cut-and-pasting links in S^3.

Dehn Surgery

Dehn surgery constructs closed, oriented 3-manifolds by cut-and-pasting links in S^3.

- $L = K_1 \cup \cdots \cup K_n$: an n-component link in S^3
- $(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}) \in (\mathbb{Q} \cup \{\infty\})^n$



$$M(L = \{K_1, K_2, \ldots, K_n\}; \tfrac{p_1}{q_1}, \ldots, \tfrac{p_n}{q_n}) := (S^3 - nb(L)) \cup_\phi n(S^1 \times D^2)$$
$$\phi_*([m_i]) = p_i[\mu_i] + q_i[\lambda_i] \text{ for } i = 1, \ldots, n.$$

Lickorish-Wallace Theorem

Theorem (Lickorish, Wallace, '60s)

Any closed orientable 3-manifold can be obtained by Dehn-surgery on a link in S^3.

Lickorish-Wallace Theorem

Theorem (Lickorish, Wallace, '60s)

Any closed orientable 3-manifold can be obtained by Dehn-surgery on a link in S^3.

Main Question

Which 3-manifolds can be obtained by Dehn surgery along *a knot* (single component link)?

We call such 3-manifolds *knot surgery manifolds*.

Lickorish-Wallace Theorem

Theorem (Lickorish, Wallace, '60s)

Any closed orientable 3-manifold can be obtained by Dehn-surgery on a link in S^3.

Main Question

Which 3-manifolds can be obtained by Dehn surgery along *a knot* (single component link)?

We call such 3-manifolds *knot surgery manifolds*.

Definition (Dehn-surgery number)

$DS(Y) :=$ the smallest # of components of a link that yields Y by Dehn-surgery.

c.f.) $DS(Y) = 0 \Leftrightarrow Y \cong S^3$

A necessary condition from $H_1(Y;\mathbb{Z})$

Homologies of knot surgery manifolds

If $Y = S^3_{p/q}(K)$, then $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, which is generated by the meridian class of the knot.

$$H_1(Y;\mathbb{Z}) = \begin{cases} 0 & \text{if } p = 1,\ Y : \text{homology} - S^3 \\ \mathbb{Z}/p\mathbb{Z} & \text{if } p \neq 1 \text{ or } 0,\ Y : \text{homology lens space} \\ \mathbb{Z} & \text{if } p = 0,\ Y : \text{homology} - S^1 \times S^2 \end{cases}$$

Ex) T^3 and $\mathbb{R}P^3 \# \mathbb{R}P^3$ cannot be obtained by Dehn surgery along a knot.

A necessary condition from the fundamental group

Let G be a group.
The *weight* of $G = Wt(G)$

$:= \min\{|S| : S \subset G \text{ s.t. } G = \langle S \rangle\}$

$=$ the minimum # of relations that can be added to G to make G trivial.

A necessary condition from the fundamental group

Let G be a group.
The *weight* of $G = Wt(G)$

$$:= \min\{|S| : S \subset G \text{ s.t. } G = \langle S \rangle\}$$

= the minimum # of relations that can be added to G to make G trivial.

Knot surgery obstruction from π_1

If Y is a knot surgery manifold, then $Wt(\pi_1(Y)) \leq 1$.

p.f.) The van Kampen theorem. □

$$Wt(\pi_1(Y)) \leq DS(Y)$$

Irreducibility

Theorem (Gabai, Property R, '87)

If Y is obtained by 0-surgery along a nontrivial knot, then Y is irreducible, in particular, $Y \ncong S^2 \times S^1$.

If $Y = S^3_0(K)$, then Y is irreducible or $S^2 \times S^1$.

Irreducibility

Theorem (Gabai, Property R, '87)

If Y is obtained by 0-surgery along a nontrivial knot, then Y is irreducible, in particular, $Y \ncong S^2 \times S^1$.

If $Y = S^3_0(K)$, then Y is irreducible or $S^2 \times S^1$.

Theorem (Gordon-Luecke, '89)

Any homology 3-sphere obtained by Dehn surgery on a knot is irreducible.

If $Y = S^3_{1/q}(K)$, then Y is irreducible.

Irreducibility

Theorem (Gabai, Property R, '87)

If Y is obtained by 0-surgery along a nontrivial knot, then Y is irreducible, in particular, $Y \not\cong S^2 \times S^1$.

If $Y = S^3_0(K)$, then Y is irreducible or $S^2 \times S^1$.

Theorem (Gordon-Luecke, '89)

Any homology 3-sphere obtained by Dehn surgery on a knot is irreducible.

If $Y = S^3_{1/q}(K)$, then Y is irreducible.

c.f.) $\exists$ a reducible homology lens space Y that can be obtained by knot surgery. For instance,

$$S^3_{pq}(K_{p,q}) \cong L(p,q) \# S^3_{q/p}(K)$$

for any knot K.

Summary

If Y is obtained by Dehn surgery along a knot, then

- $H_1(Y;\mathbb{Z}) \cong 0$, $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}$,
- $Wt(\pi_1(Y)) = 1$,
- If Y is a homology-S^3 ($H_1 \cong 0$), then Y is irreducible.
  If Y is a homology-$S^2 \times S^1$ ($H_1 \cong \mathbb{Z}$), then Y is irreducible or $S^2 \times S^1$.

Summary

If Y is obtained by Dehn surgery along a knot, then

- $H_1(Y;\mathbb{Z}) \cong 0$, $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}$,
- $Wt(\pi_1(Y)) = 1$,
- If Y is a homology-S^3 ($H_1 \cong 0$), then Y is irreducible.
  If Y is a homology-$S^2 \times S^1$ ($H_1 \cong \mathbb{Z}$), then Y is irreducible or $S^2 \times S^1$.

Main Question

Are these conditions sufficient for Y to be a knot surgery manifold?

Homology 3-spheres

Theorem (Auckly '97, Kirby problem 3.6.(C).)

There exists an irreducible $\mathbb{Z}HS^3$ that is not a Dehn-surgery along a knot.

- used Taubes' periodic ending theorem.
- It is not known that the weight of π_1 of this example is 1 or not.

Homology 3-spheres

Theorem (Auckly '97, Kirby problem 3.6.(C).)

There exists an irreducible $\mathbb{Z}HS^3$ that is not a Dehn-surgery along a knot.

- used Taubes' periodic ending theorem.
- It is not known that the weight of π_1 of this example is 1 or not.

Theorem (Hom-Karakurt-Lidman, '16)

Let $Y_p = \Sigma(p, 2p-1, 2p+1)$ and p be an even integer ≥ 8
$\Rightarrow Y_p$: integral homology Seifert-fibered spaces, $Wt(\pi_1(Y)) = 1, DS(Y_p) = 2$

- used Heegaard Floer theory.

Homology 3-spheres

Theorem (Auckly '97, Kirby problem 3.6.(C).)

There exists an irreducible $\mathbb{Z}HS^3$ that is not a Dehn-surgery along a knot.

- used Taubes' periodic ending theorem.
- It is not known that the weight of π_1 of this example is 1 or not.

Theorem (Hom-Karakurt-Lidman, '16)

Let $Y_p = \Sigma(p, 2p-1, 2p+1)$ and p be an even integer ≥ 8
$\Rightarrow Y_p$: integral homology Seifert-fibered spaces, $Wt(\pi_1(Y)) = 1, DS(Y_p) = 2$

- used Heegaard Floer theory.

Open question: Is there a homology 3-sphere with $Wt(\pi_1) \geq 2$?
A conjecture of Wiegold:
Does every finitely presented perfect group have weight 1?

Homology lens spaces

Theorem (Boyer-Lines, '90)

There exist infinitely many irreducible homology lens spaces such that $Wt(\pi_1) = 1$ *and but they are NOT knot surgery manifolds.*

- These are Seifert-fibered manifolds.
- used the surgery formula for Casson's invariant.

Homology lens spaces

Theorem (Boyer-Lines, '90)

There exist infinitely many irreducible homology lens spaces such that $Wt(\pi_1) = 1$ *and but they are NOT knot surgery manifolds.*

- These are Seifert-fibered manifolds.
- used the surgery formula for Casson's invariant.

Theorem (Hoffman-Walsh, '15)

There exist infinitely many irreducible hyperbolic homology lens spaces such that $Wt(\pi_1) = 1$ *and but they are not knot surgery manifolds.*

Homology lens spaces

Theorem (Boyer-Lines, '90)

There exist infinitely many irreducible homology lens spaces such that $Wt(\pi_1) = 1$ and but they are NOT knot surgery manifolds.

- These are Seifert-fibered manifolds.
- used the surgery formula for Casson's invariant.

Theorem (Hoffman-Walsh, '15)

There exist infinitely many irreducible hyperbolic homology lens spaces such that $Wt(\pi_1) = 1$ and but they are not knot surgery manifolds.

c.f) (Sato-Taniguchi, '18) There exist infinitely many irreducible (non-Seifert fibered) toroidal homology lens spaces that are NOT knot surgery manifolds. It is still open if theirs have weight one π_1.

Main Question

What about 3-manifolds with $H_1(Y) \cong \mathbb{Z}$?

Question (Aschenbrenner-Friedl-Wilton, '16)

Suppose Y is an irreducible, closed, oriented 3-manifold such that

$$b_1(Y) = Wt(\pi_1(Y)) = 1,$$

Then is Y obtained by Dehn-surgery along a knot?

Goal: to answer this question *negatively* by presenting two different families of examples.

Example I (Non-Seifert fibered examples)

Theorem (Hedden-Kim-Mark-P., '17)

There is a family of closed oriented 3-manifolds, $\{M_k\}_{k\geq 1}$ such that

- M_k*: irreducible,* $H_1(M_k;\mathbb{Z}) \cong \mathbb{Z}$*, and* $Wt(\pi_1(M_k)) = 1$
- M_k *cannot be obtained by Dehn-surgery on a knot.*

Example I (Non-Seifert fibered examples)

Theorem (Hedden-Kim-Mark-P., '17)

There is a family of closed oriented 3-manifolds, $\{M_k\}_{k\geq 1}$ such that

- M_k*: irreducible,* $H_1(M_k;\mathbb{Z}) \cong \mathbb{Z}$*, and* $Wt(\pi_1(M_k)) = 1$
- M_k *cannot be obtained by Dehn-surgery on a knot.*

- M_k *is not homology cobordant to any knot surgery manifold.*
- M_k *is not homology cobordant to any Seifert-fibered 3-manifold.*
- M_k *is not homology cobordant to* M_l *if* $k \neq l$*.*

Two closed, oriented 3-manifolds Y_1 and Y_2 are *homology cobordant* if $\exists$ a smooth oriented cobordism W between them for which the inclusion maps induce isomorphisms, $i_* : H_*(Y_i;\mathbb{Z}) \to H_*(W;\mathbb{Z})$.

Example II (Seifert fibered examples)

Theorem (Hedden-Kim-Mark-P., '17)

There is a family of closed oriented 3-manifolds, $\{N_k\}_{k\geq 1}$ such that

1. N_k*: irreducible,* $H_1(N_k;\mathbb{Z}) \cong \mathbb{Z}$*, and* $Wt(\pi_1(N_k)) = 1$
2. N_k *is a Seifert manifold over* S^2 *with three exceptional fibers.*
3. *If* k *is odd,* N_k *is not obtained by Dehn surgery on a knot in* S^3*.*

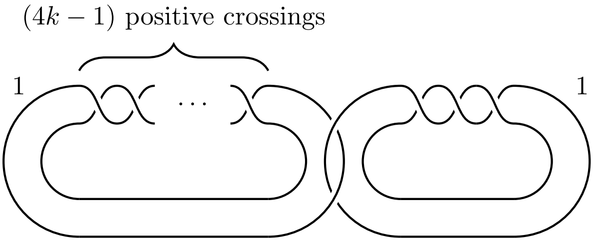
Example II (Seifert fibered examples)

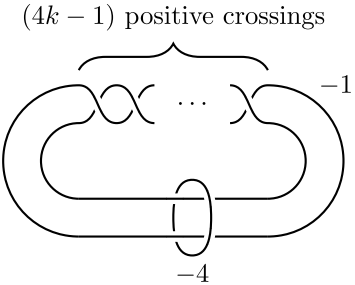
Theorem (Hedden-Kim-Mark-P., '17)

There is a family of closed oriented 3-manifolds, $\{N_k\}_{k\geq 1}$ *such that*

1. N_k*: irreducible,* $H_1(N_k;\mathbb{Z}) \cong \mathbb{Z}$*, and* $Wt(\pi_1(N_k)) = 1$
2. N_k *is a Seifert manifold over* S^2 *with three exceptional fibers.*
3. *If* k *is odd,* N_k *is not obtained by Dehn surgery on a knot in* S^3*.*
4. *If* k *is odd,* N_k *is not homology cobordant to Dehn surgery along a knot in* S^3*.*

The manifolds M_k and N_k

$M_k :=$ 

$N_k :=$ 

Topological properties of splices

Given two oriented 3-manifolds with torus boundary, Y_1 and Y_2,

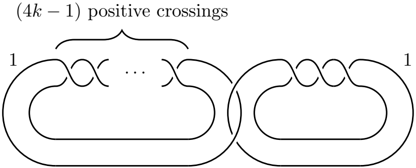
$$\text{A } \textit{splice} \text{ of } Y_1 \ \& \ Y_2 = Y_1 \cup_T Y_2$$

Proposition

Let Y_1, Y_2 be irreducible manifolds, each with an incompressible torus as boundary. Then any splice of Y_1 and Y_2 is irreducible.

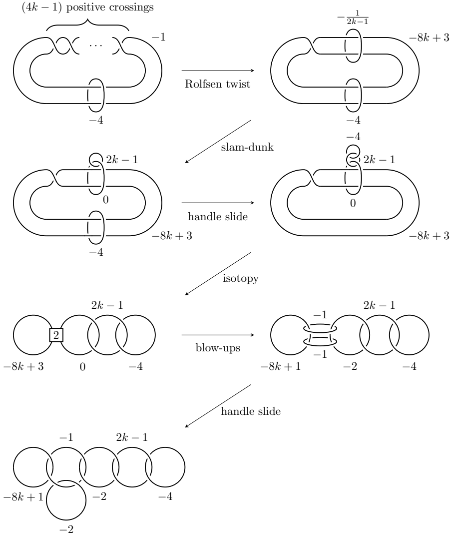
Proposition

Any splice of complements of knots in the 3-sphere has weight one fundamental group.



M_k is irreducible, and $Wt(\pi_1(M_k)) = 1$

Seifert fibering structure of N_k



Note that any orientable, reducible Seifert fibered 3-manifold over S^2 is homeomorphic to either $S^1 \times S^2$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$.

Hence N_k is irreducible.

Weight of $\pi_1(N_k)$

$$\begin{aligned}\pi_1(N_k) &\cong \langle x_1, x_2, x_3, h \mid x_1^{16k-2} = h^{8k-3}, x_2^{8k-1} = h, x_3^2 = h, x_1x_2x_3 = h, [h, x_i] = 1\rangle \\ &\cong \langle x_1, x_2, h \mid x_1^{16k-2} = h^{8k-3}, x_2^{8k-1} = h, h = x_1x_2x_1x_2, [h, x_1] = [h, x_2] = 1\rangle \\ &\cong \langle x_1, x_2 \mid x_1^{16k-2} = x_2^{(8k-1)(8k-3)}, x_2^{8k-1} = (x_1x_2)^2, [x_2^{8k-1}, x_1] = 1\rangle.\end{aligned}$$

$$\begin{aligned}\pi_1(N_k)/\langle\langle x_2^{4k-2}x_1^{-1}\rangle\rangle &\cong \langle x_1, x_2 \mid x_1^{16k-2} = x_2^{(8k-1)(8k-3)}, x_2^{8k-1} = (x_1x_2)^2, x_1 = x_2^{4k-2} \\ &\cong \langle x_2 \mid x_2^{(8k-1)(8k-4)} = x_2^{(8k-1)(8k-3)}, x_2^{8k-1} = x_2^{8k-2}\rangle \\ &\cong \langle x_2 \mid x_2^{(8k-1)(8k-4)} = x_2^{(8k-1)(8k-3)}, x_2 = 1\rangle = 1.\end{aligned}$$

Heegaard Floer homology

- Y: a closed oriented 3-manifold, $\mathfrak{t}$: a Spin^c structure over Y

Heegaard Floer homology (Ozsváth-Szabó, 2000)

$$(Y, \mathfrak{t}) \longmapsto HF^\circ(Y, \mathfrak{t}),\ \circ \in \{-, \infty, +\}$$

- HF°: relative $\mathbb{Z}$-graded (Maslov grading) modules over $\mathbb{F}[U]$, $deg(U) = -2$.
- Y
  $\rightsquigarrow$ a pointed Heegaard diagram $(\Sigma_g, \{\alpha_1, \ldots, \alpha_g\}, \{\beta_1, \ldots, \beta_g\}, w)$ for Y
  $\rightsquigarrow$ Heegaard Floer homology $\approx$ Lagrangian Floer homology for $(Sym^g(\Sigma), T_\alpha := [\alpha_1 \times \cdots \times \alpha_g], T_\beta := [\beta_1 \times \cdots \times \beta_g])$
- CF° is generated by $T_\alpha \cap T_\beta$ and ∂ counts the J-holomorphic disks connecting them.
- U. lowers the grading by 2.

Heegaard Floer correction terms

If $\mathfrak{t}$ is torsion ($c_1(\mathfrak{t})$ is torsion), then relative Maslov grading can be lifted Absolute $\mathbb{Q}$-grading.

If $b_1(Y) = 0$, then

$$HF^+(Y, \mathfrak{t}) \cong \mathcal{T}^+_{d(Y,\mathfrak{t})} \oplus HF^{red}(Y, \mathfrak{t}),$$

where $\mathcal{T}^+_d \cong \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$ of which 1 has grading $d \in \mathbb{Q}$.

Heegaard Floer correction terms

If $\mathfrak{t}$ is torsion ($c_1(\mathfrak{t})$ is torsion), then relative Maslov grading can be lifted Absolute $\mathbb{Q}$-grading.

If $b_1(Y) = 0$, then

$$HF^+(Y, \mathfrak{t}) \cong \mathcal{T}^+_{d(Y,\mathfrak{t})} \oplus HF^{red}(Y, \mathfrak{t}),$$

where $\mathcal{T}^+_d \cong \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$ of which 1 has grading $d \in \mathbb{Q}$.

If Y is a homology $S^2 \times S^1$,

$$HF^+(Y, \mathfrak{t}_0) \cong \mathcal{T}^+_{d_{-\frac{1}{2}}(Y,\mathfrak{t})} \oplus \mathcal{T}^+_{d_{+\frac{1}{2}}(Y,\mathfrak{t})} \oplus HF^{red}(Y, \mathfrak{t}).$$

where $\mathfrak{t}_0$ is the spin^c structure with $c_1(\mathfrak{t}_0) = 0$.

Heegaard Floer correction terms

If $\mathfrak{t}$ is torsion ($c_1(\mathfrak{t})$ is torsion), then relative Maslov grading can be lifted Absolute $\mathbb{Q}$-grading.

If $b_1(Y) = 0$, then

$$HF^+(Y, \mathfrak{t}) \cong \mathcal{T}^+_{d(Y,\mathfrak{t})} \oplus HF^{red}(Y, \mathfrak{t}),$$

where $\mathcal{T}^+_d \cong \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$ of which 1 has grading $d \in \mathbb{Q}$.

If Y is a homology $S^2 \times S^1$,

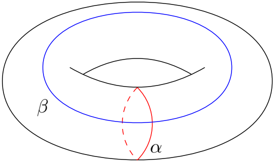
$$HF^+(Y, \mathfrak{t}_0) \cong \mathcal{T}^+_{d_{-\frac{1}{2}}(Y,\mathfrak{t})} \oplus \mathcal{T}^+_{d_{+\frac{1}{2}}(Y,\mathfrak{t})} \oplus HF^{red}(Y, \mathfrak{t}).$$

where $\mathfrak{t}_0$ is the spin^c structure with $c_1(\mathfrak{t}_0) = 0$.

We call $d(Y, \mathfrak{t})$ is called the *correction term* or *d-invariant* of $(Y, \mathfrak{t})$.
Note that this is analogous to Frøyshov's invariants in Seiberg-Witten theory.

Examples

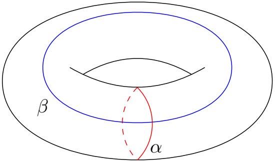
Example) S^3



$$HF^+(S^3) \cong \mathbb{Z}[U, U^{-1}]/UZ[U] =: \mathcal{T}_0^+$$

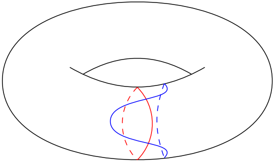
Examples

Example) S^3



$$HF^+(S^3) \cong \mathbb{Z}[U, U^{-1}]/UZ[U] =: \mathcal{T}_0^+$$

Example) $S^1 \times S^2$



$$HF^+(S^2 \times S^1, \mathfrak{t}_0) \cong \mathcal{T}^+_{-\frac{1}{2}} \oplus \mathcal{T}^+_{+\frac{1}{2}}$$

Dehn-surgery obstruction

Theorem (Ozsváth-Szabó, '03)

Suppose Y is a $\mathbb{Q}HS^3$, bounding a negative definite 4-manifold W. Then for each spin^c structure $\mathfrak{s}$ over W,

$$c_1(\mathfrak{s})^2 + b_2(W) \leq 4d(Y, \mathfrak{s}|_Y),$$

where $c_1(-)$ is the first Chern class.

Dehn-surgery obstruction

Theorem (Ozsváth-Szabó, '03)

Suppose Y is a $\mathbb{Q}HS^3$, bounding a negative definite 4-manifold W. Then for each spin^c structure $\mathfrak{s}$ over W,

$$c_1(\mathfrak{s})^2 + b_2(W) \leq 4d(Y, \mathfrak{s}|_Y),$$

where $c_1(-)$ is the first Chern class.

p.f.)

$$\begin{array}{ccc} HF^{\infty}(S^3) & \xrightarrow{F_W^{\infty}(\cong)} & HF^{\infty}(Y) \\ \downarrow \pi & & \downarrow \pi \\ HF^{+}_{d(Y)-deg(F)}(S^3) & \xrightarrow{F_W^{+}} & HF^{+}_{d}(Y) \end{array}$$

$\Rightarrow d(Y) - deg(F_W) \geq 0$ □

0-surgery obstruction

If Y is obtained by 0-surgery on a knot, then Y bounds
$W = B^4 \cup (\text{0-framed 2-handle along } K)$

0-surgery obstruction

If Y is obtained by 0-surgery on a knot, then Y bounds $W = B^4 \cup$ (0-framed 2-handle along K)

Corollary (0-surgery obstruction)

If Y is obtained by 0-surgery on a knot, then

$$d_{\frac{1}{2}}(Y) \leq \frac{1}{2} \quad \text{and} \quad d_{-\frac{1}{2}}(Y, \mathfrak{t}_0) \geq -\frac{1}{2}.$$

0-surgery obstruction

If Y is obtained by 0-surgery on a knot, then Y bounds $W = B^4 \cup$ (0-framed 2-handle along K)

Corollary (0-surgery obstruction)

If Y is obtained by 0-surgery on a knot, then

$$d_{\frac{1}{2}}(Y) \leq \frac{1}{2} \quad \textit{and} \quad d_{-\frac{1}{2}}(Y, \mathfrak{t}_0) \geq -\frac{1}{2}.$$

Proposition

If Y is a Seifert fibered 3-manifold with $b_1(Y) = 1$, then

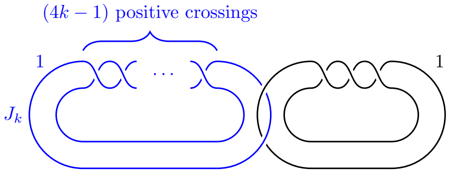
$$d_{\frac{1}{2}}(Y) \leq \frac{1}{2} \quad \textit{and} \quad d_{-\frac{1}{2}}(Y, \mathfrak{t}_0) \geq -\frac{1}{2}.$$

The correction terms of M_k

Theorem (HKMP)

For any $k \geq 1$,

$$d_{\frac{1}{2}}(M_k) = -2k + \frac{1}{2} \quad \textit{and} \quad d_{-\frac{1}{2}}(M_k) = -\frac{5}{2}.$$

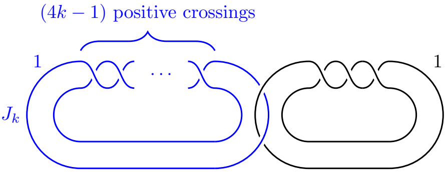


The correction terms of M_k

Theorem (HKMP)

For any $k \geq 1$,

$$d_{\frac{1}{2}}(M_k) = -2k + \frac{1}{2} \quad \textit{and} \quad d_{-\frac{1}{2}}(M_k) = -\frac{5}{2}.$$



Consider M_K as the 1-surgery of $\Sigma(2,3,5)$ along the knot J_k.

1. Understand the knot Floer homology of $(S^3, T_{2,4k-1})$ and (Σ, μ).
2. Using the Künneth formula type formula of the knot Floer homolgy to have the knot Floer homology of

$$(\Sigma, J_k) = (S^3, T_{2,4k-1}) \# (\Sigma, \mu).$$

3. By using the surgery formula, we understand HF^+ of M_k. □

Rohlin invariants

For $(Y, \mathfrak{s})$ be a spin 3-manifold. The *Rohlin invariant*, $\mu(Y; \mathfrak{s}) \in \mathbb{Q}/2\mathbb{Z}$

$$\mu(Y; \mathfrak{s}) \colon = \frac{\sigma(W)}{8} \bmod 2,$$

where W is a compact 4-manifold bounded by Y that admits a spin structure extending $\mathfrak{s}$.
Note that if Y is a homology-$S^1 \times S^2$, then Y admits two spin structures, say $\mathfrak{s}_0$ and $\mathfrak{s}_1$.

Proposition

Let Y be a homology sphere and $K \subset Y$ a knot. Then

$$\mu(Y_0(K), \mathfrak{s}_0) = \mu(Y)$$

and

$$\mu(Y_0(K), \mathfrak{s}_1) = \mu(Y) + Arf(K).$$

Rohlin invariants of N_k

Corollary

If both $\mu(Y, \mathfrak{s}_0)$ and $\mu(Y, \mathfrak{s}_1)$ are nonzero, then Y cannot be obtained by 0 surgery along a knot in S^3.

One compute $\mu(N_k, \mathfrak{s}_i) \equiv k$ (modulo 2) for $i = 0, 1$ by using plumbing diagrams P_k. Note that

$$\mu(N_k, \mathfrak{s}_i) \equiv \frac{1}{8}(\sigma(P_k) - \nu_i.\nu_i) \mod 2,$$

where $\nu_1, \nu_2 \in H_2(P_k; \mathbb{Z}/2)$ are represented by embedded spheres or a disjoint union thereof, satisfying $\nu_i \cdot x \equiv x \cdot x \mod 2$ for each homology class x.

$$\mu(N_k, \mathfrak{s}_i) \equiv k \mod 2 \text{ for } i = 0, 1.$$

Hence N_k is NOT a knot surgery manifold for k odd.

Questions

Question

Is there a such hyperbolic example?

Since d-invariant and Rohlin invariant are homology cobordant invariants and any 3-manifold is homology cobordant to a hyperbolic 3-manifolds, we know that there is a hyperbolic 3-manifold with $b_1 = 1$ and $DS > 1$. What about $Wt(\pi_1)$?

Questions

Question

Is there a such hyperbolic example?

Since d-invariant and Rohlin invariant are homology cobordant invariants and any 3-manifold is homology cobordant to a hyperbolic 3-manifolds, we know that there is a hyperbolic 3-manifold with $b_1 = 1$ and $DS > 1$. What about $Wt(\pi_1)$?

Question

Is there an irreducible 3-manifold Y such that $Wt(\pi_1(Y)) \leq 2$ and $DS(Y) \geq 3$? More generally, is there an irreducible 3-manifold s.t. $DS(Y) - Wt(\pi_1(Y))$ can be arbitrarily large?

Is there a non-trivial obstruction on a 3-manifold being a Dehn-surgery along n-components link?