Instantons and Uniformizations of 4-orbifolds

East Asian Conference on Gauge theory and Related topics

Kyoto Univ. Sept. 15, 2018

(Revised in Sept. 26, 2018)

Yoshihiro FUKUMOTO

Ritsumeikan Univ.

Outline

$\boxed{1}$ Motivation

- $\mathbb{Q}$-homology cobordism among lens spaces
- Uniformization of orbifolds

$\boxed{2}$ Main Theorem

- Negative-definite cobordism among lens sps
- Finite Uniformization of 4-orbfds

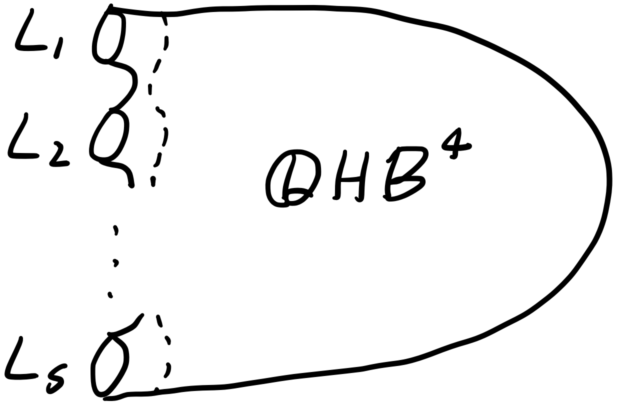
$\boxed{3}$ Idea of Proof.

- Instanton moduli spaces

$\boxed{4}$ Concluding Remark

$\boxed{1}$ Motivation ① $\mathbb{Q}$-homology cobordism among lens spaces

P. Lisca (2007) classified connected sums of lens spaces which smoothly bound $\mathbb{Q}HB^4$



$\Leftarrow$

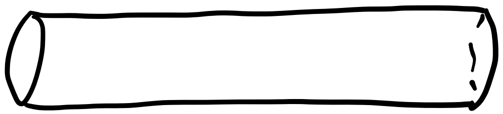
Donaldson's Thm A

X : closed smooth 4-mfd

X : neg. def. $\Longrightarrow$

$Q_X \cong \langle -1 \rangle^N$

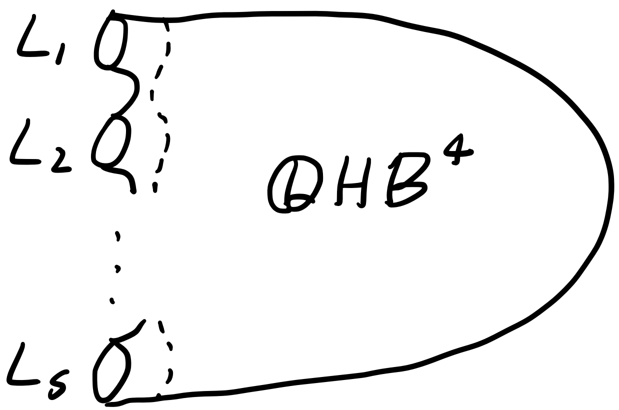
Ex. $L(a,-b)$



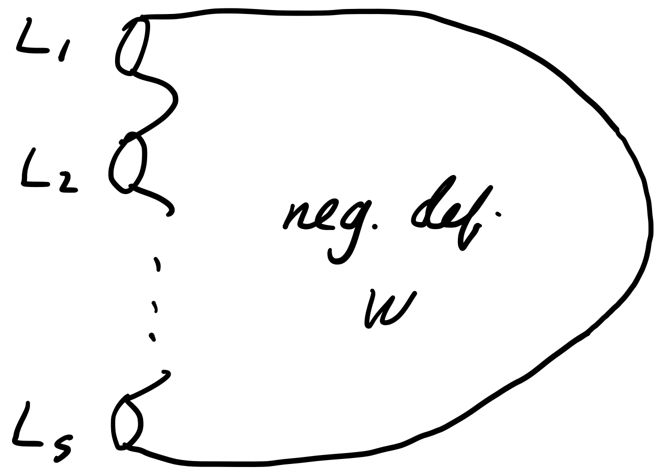
$L(a,b)$

$[0,1] \times L(a,b)$

Question: What can we say if we replace



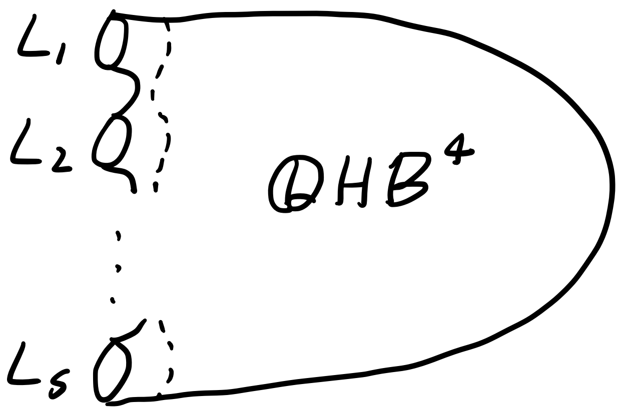
with



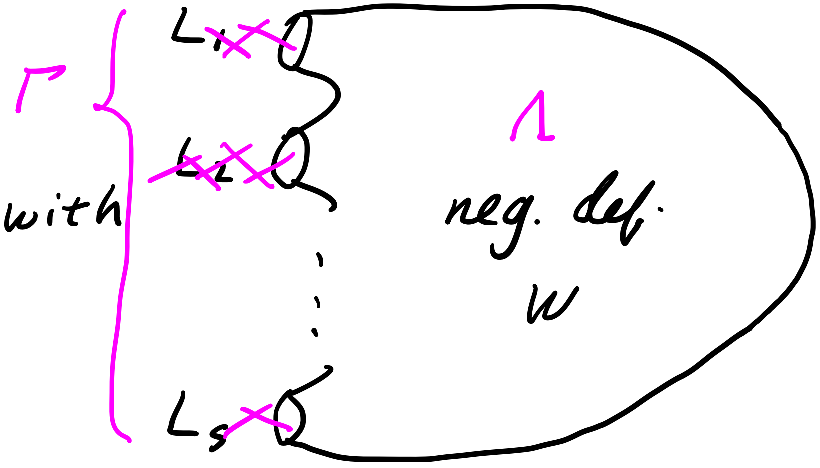
?

$$\bigoplus_{i=1}^{s} H_*(L_i;\mathbb{Z}) \xrightarrow{i_*} H_*(W;\mathbb{Z})$$

Question: What can we say if we replace



with



?

$$\bigoplus_{i=1}^{s} H_*(L_i;\mathbb{Z}) \xrightarrow{i_*} H_*(W;\mathbb{Z})$$

Remark

[D.H. Choe - K. Park (2018)]

FINITENESS result for $\Gamma \oplus \Lambda \hookrightarrow \langle -1 \rangle^{rk(\Gamma)+rk(\Lambda)}$

$\det(\Lambda) = D$

$\delta(\Lambda) \leq C$

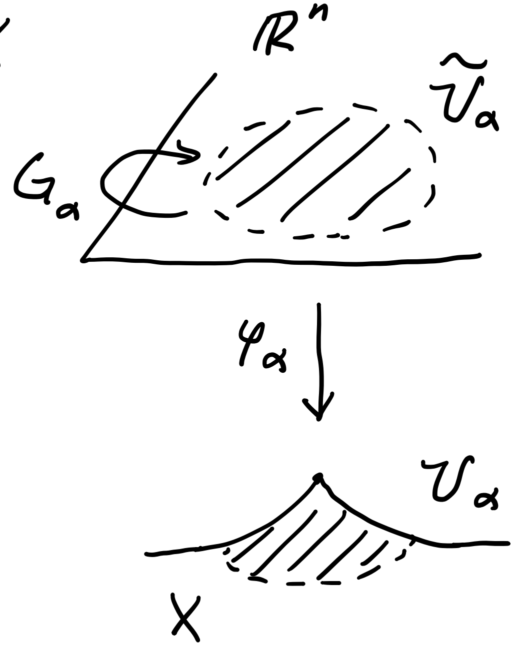
$\boxed{1}$ Motivation ② Uniformization Problem

Orbifolds (**V-mfds**) [I. Satake (1956), W. Thurston (1981)]

X : n-dim. smooth orbfd

X : Hausdorff sp., $X = \bigcup_\alpha U_\alpha$ $\quad U_\alpha \underset{\text{open}}{\subset} X$

$\{\tilde{U}_\alpha, G_\alpha, \varphi_\alpha\}$: local uniformizing system



$$\begin{tikzcd} \underset{\text{finite}}{G_\alpha} \overset{\text{eff.}}{\curvearrowright} \tilde{U}_\alpha \underset{\text{open}}{\subset} \mathbb{R}^n \arrow[d] \arrow[dr, "\varphi_\alpha"] \\ \tilde{U}_\alpha/G_\alpha \approx U_\alpha \end{tikzcd} \quad C^\infty ,$$

$$\begin{tikzcd} \tilde{U}_\alpha \arrow[r, hook, "\lambda_{\beta\alpha}", "C^\infty"'] \arrow[d, "\varphi_\alpha"'] & \tilde{U}_\beta \arrow[d, "\varphi_\beta"] \\ U_\alpha \subset & U_\beta \end{tikzcd}$$

$|\Sigma X| := \{x \in X \mid \exists \alpha, \exists g \in G_\alpha \setminus \{e\}, \exists \tilde{x} \in \tilde{U}_\alpha^g, \varphi_\alpha(\tilde{x}) = x\}$

Ex. $\widetilde{X}$: smooth n-mfd (finite) $\pi \curvearrowright \widetilde{X}$: prop. disconti.

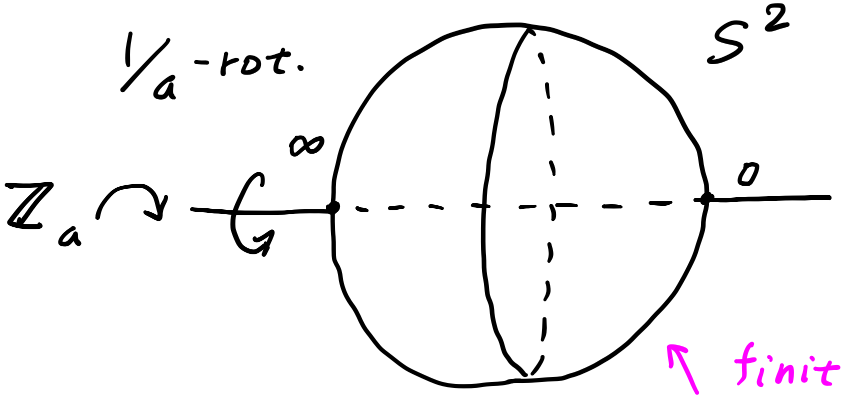
$\Rightarrow X = \widetilde{X}/\pi$: smooth n-orbfd

$(\widetilde{X}, \pi)$: (finite) uniformization of X

Rmk : might be

$$X \overset{\nexists}{=} \widetilde{X}/\pi$$

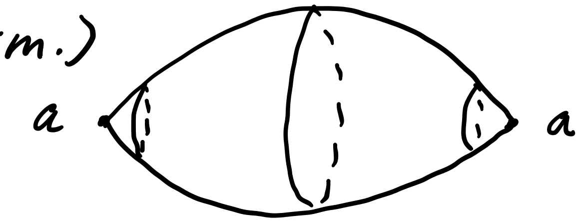
"Bad orbfd"



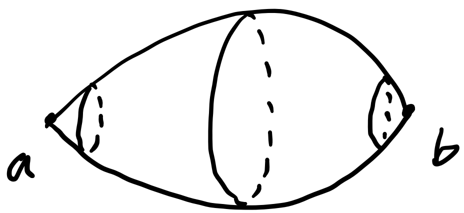
finite uniformization of $S^2(a,a)$

$\downarrow$ $\nexists \downarrow$ $\nexists \downarrow$

"Spindle" (2-dim.)

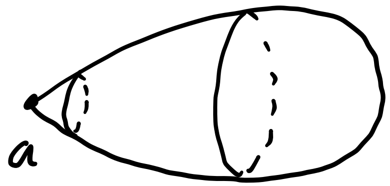


$X = S^2/\mathbb{Z}_a = S^2(a,a)$



$S^2(a,b)$ $(a \neq b)$

"Teardrop"



$S^2(a)$

1 Motivation ② Finite Uniformization Problem (2)

Problem [c.f. M. Kato, M. Namba]

> Let X be an orbfd.
> Find a "good" condition on X for the existence of a finite uniformization of X

"good" $\leftarrow$ without refering to $\pi_1(X \setminus |\Sigma X|)$

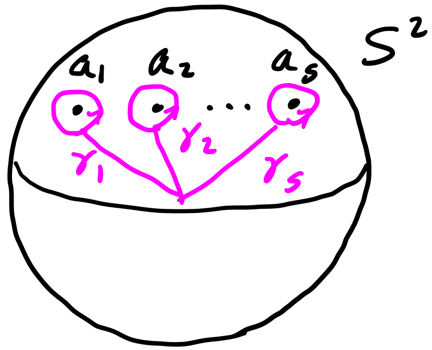
c.f. M. Kato "b-completeness" (1986)

M. Namba "Fenchel's Problem" (1987)

[R. Fox (1952) c.f. S. Bundgaad - J. Nielsen (1951)]

$X = S^2(a_1, \cdots, a_s)$

$s \geq 3 \Longrightarrow \exists$ fin. unif. $(\tilde{X}, G)$ of X, $X \cong \tilde{X}/G$



$\because$ $\pi_1^{orb}(X) = \langle \gamma_1, \cdots, \gamma_s \mid \gamma_1^{a_1} = \cdots = \gamma_s^{a_s} = \gamma_1 \cdots \gamma_s = 1 \rangle$

$\underset{\exists \neq 1}{\longrightarrow}$ (permutation group)

③ Key Example "4-dim. Spindle"

$$\mathbb{Z}_a \curvearrowright S^4 = (\mathbb{C} \oplus \mathbb{C}) \cup \{\infty\} \qquad \zeta_a = e^{2\pi i/a}$$

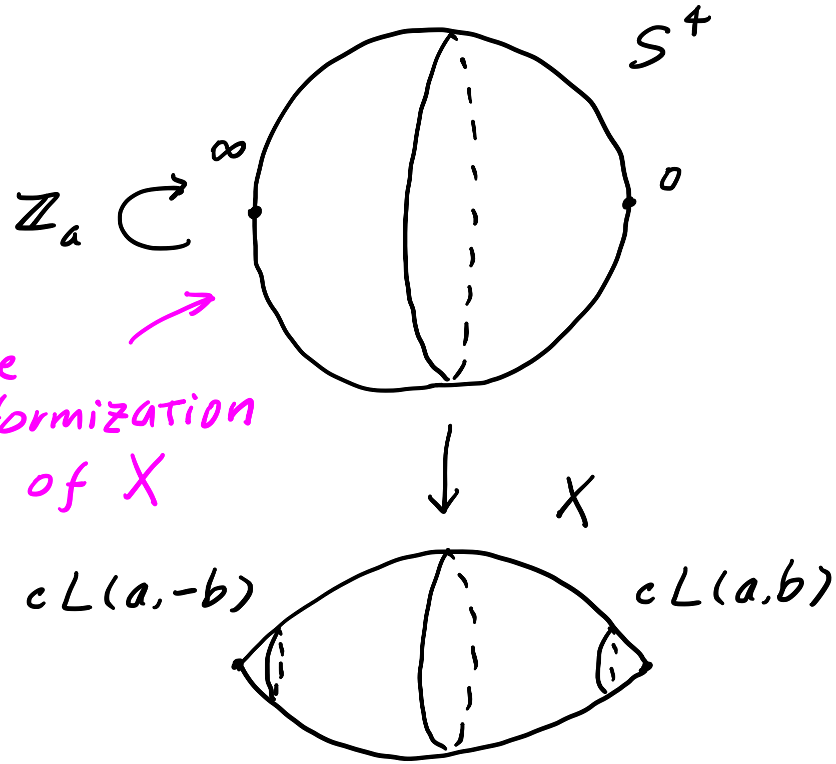
$$\zeta_a \cdot (z, w) = (\zeta_a z, \zeta_a^b w) \qquad a, b : \text{coprime}$$

$X = S^4/\mathbb{Z}_a$: smooth 4-orbfd

$|\Sigma X| = \{[0], [\infty]\}$

$U_{[0]} \approx cL(a, b)$

$U_{[1]} \approx cL(a, -b)$

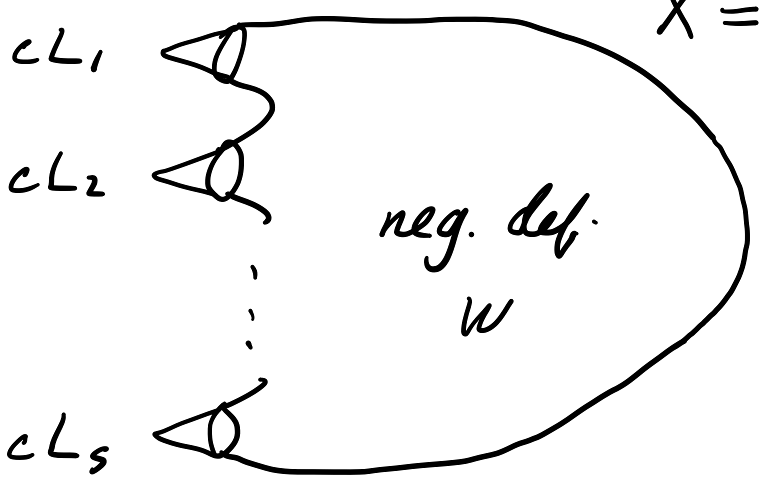


Question: Does the following 4-orbfd have a finite Uniformization?

$$\exists ?\ \pi \circlearrowright \tilde{X}$$

$$\downarrow$$

$$X \overset{?}{=} \tilde{X}/\pi$$



I don't KNOW. But,

$\boxed{\text{Instantons KNOW* finite Uniformizations!}}$

* a little

2 Main Theorem

X : closed ori. smooth 4-orbfd, $|\Sigma X| = \{p, p'\}$

$U_p \approx cL(a,-b)$ $(b>0,\ a \gg b)$, $U_{p'} \approx cL(a',b')$ $(a' \gg |b'|)$

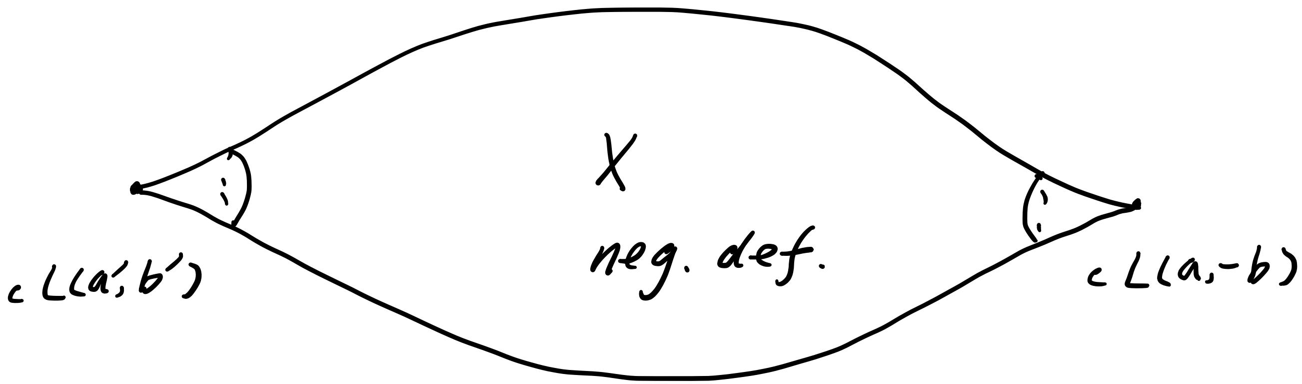
① $b_2^+(X) = 0$, $H_1(X;\mathbb{Z}) = 0$

② $a \geq a'$, $\dfrac{b}{a} \leq \dfrac{|b'|}{a'}$, a : odd, b : prime

③ $\mu = 0$ $(\star)$

$\Rightarrow$ $U_{p'} \approx cL(a,b)$

$\mathbb{Z}_a \curvearrowright \exists \tilde{X}$: closed smooth 4-mfd s.t. $\tilde{X}/\mathbb{Z}_a \cong X$



Remark

① $a \gg |b| \Longleftarrow a > b^2$

② If b is not prime, $\mathcal{U}_{p'} \approx cL(a, \tilde{b})$

$$(b = l_1' l_2',\ l_2' \tilde{b} \equiv l_1' \bmod a)$$

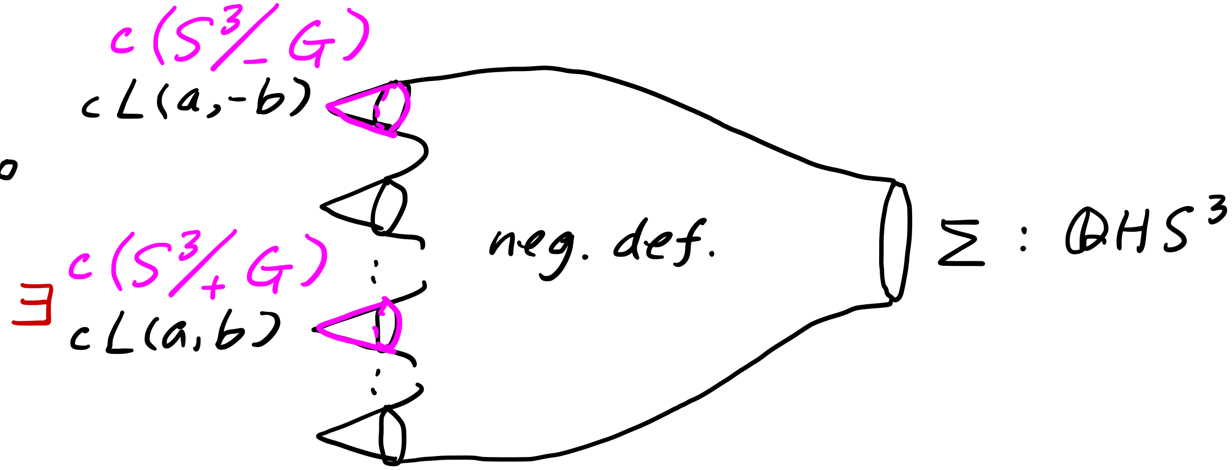
③ $\mathcal{U}_p \approx cL(a, +b) \not\Longrightarrow \mathcal{U}_{p'} \approx cL(a, -b)$

$\Gamma = \underset{-2}{\bullet} - \underset{-2}{\bullet} - \cdots - \underset{-2}{\bullet}$ $\quad \partial P(\Gamma) = L(m, -1),\ X = \widehat{P(\Gamma)}$

④ $\mu = \# \left\{ e \in H^2(X \setminus \{p\} : \mathbb{Z}) \;\middle|\; e^2 = -\frac{b}{a},\ i_p^* e = \pm e(S^3 \underset{\mathbb{Z}_a}{\times} S^1) \right\} / \{\pm 1\}$

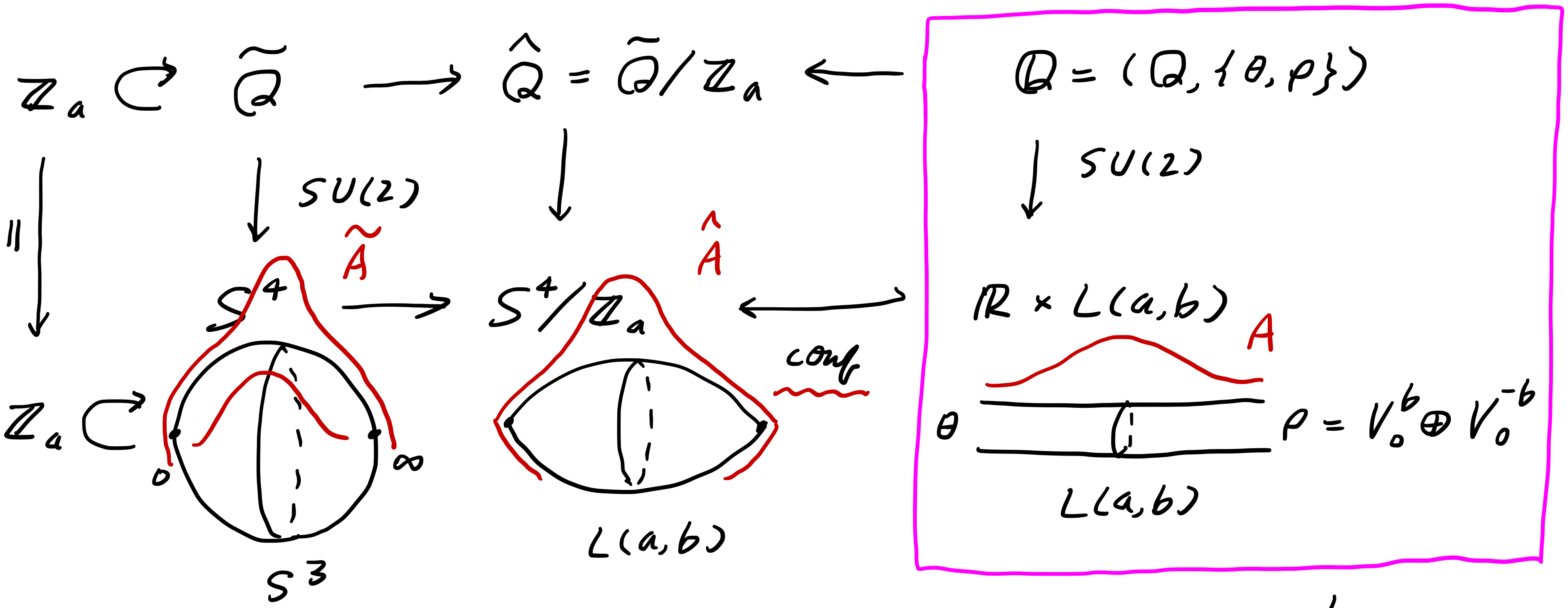
$= \#$ of reducible instantons

⑤ Can be generalized to



$\boxed{3}$ Idea of pf. [c.f. M. Furuta (1990)] $S^1 \supset \mathbb{Z}_a \circlearrowright V_0 = \mathbb{C}$

⓪ $\mathbb{Z}_a$-invariant instanton over $S^4 = (V_0 \oplus V_0^{-b}) \cup \{\infty\}$



$$c_2(\mathbb{Q}) = c_2(\hat{Q})[S^4/\mathbb{Z}_a] = \frac{1}{a} c_2(\tilde{Q})[S^4] = \frac{b}{a}$$

$$\mathcal{M}(\mathbb{Q}) = \{A \in \mathcal{A}(\mathbb{Q}) \mid F_A^+ = 0\} / \mathcal{G}(\mathbb{Q})$$

$$\cong \mathcal{M}(\tilde{Q}, \mathbb{Z}_a) \cong \mathbb{R}_+$$

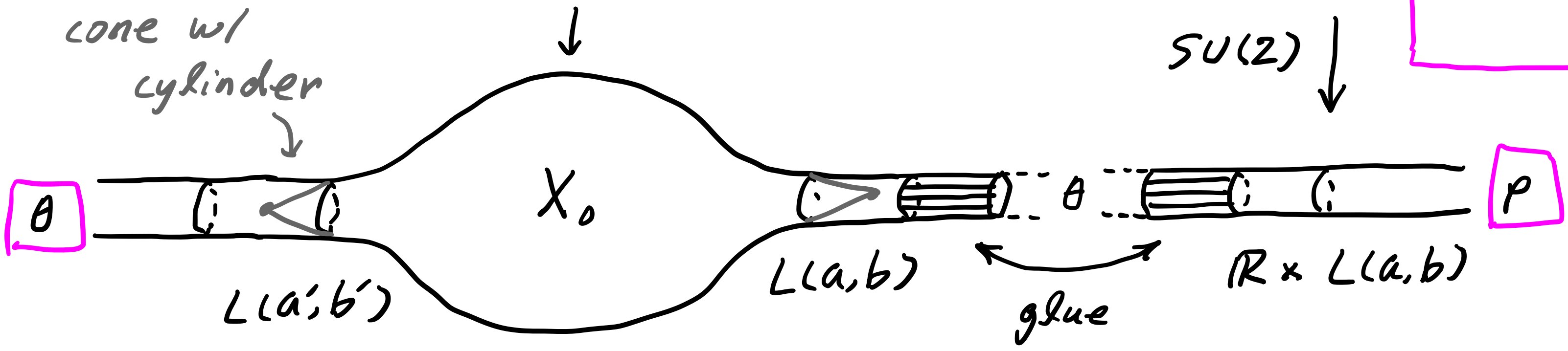
provided $b > 0$, $a \gg b$

① Instantons on 4-manifold with cylindrical ends

$\mathbb{P} :=$ replace cone w/ cylinder

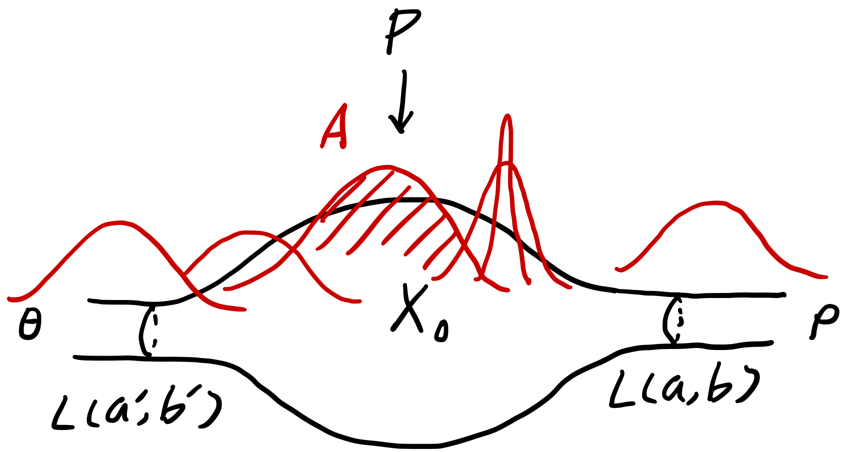
$\boxed{P} := X_0 \times SU(2) \quad \# \quad Q$

$\boxed{c_2(\mathbb{P}) = c_2(\mathbb{Q}) = \frac{b}{a}}$



$$\mathcal{M}(\mathbb{P}) := \{ A \in \mathcal{A}(\mathbb{P}) \mid F_A^+ = 0 \} / \mathcal{G}(\mathbb{P}) \qquad (g : \text{Riem. met. on } X_0)$$

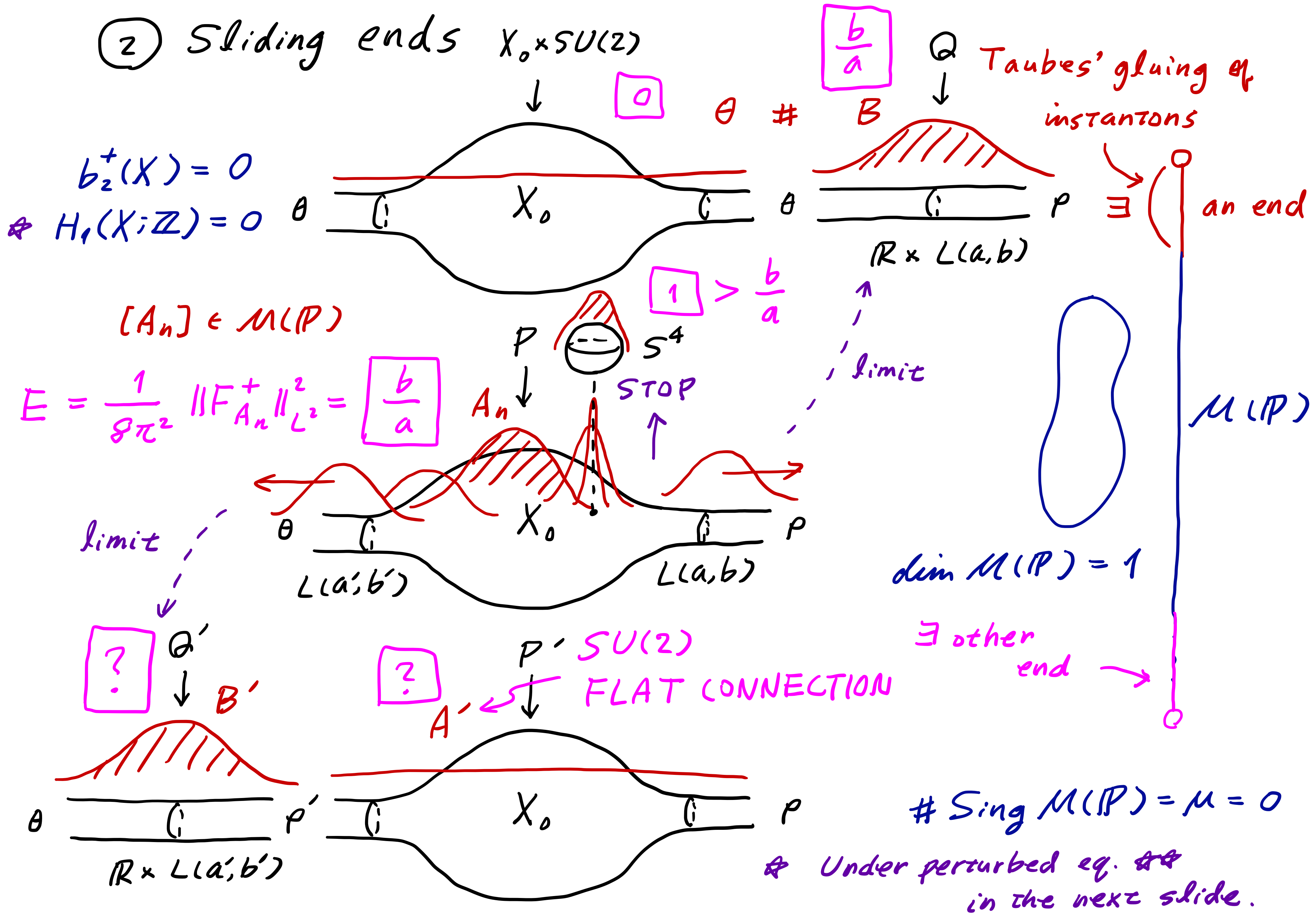
$\sim$ the moduli space of instantons on $\mathbb{P}$



$$F_A^+ = \frac{1}{2}(F_A + *_g F_A) = 0$$

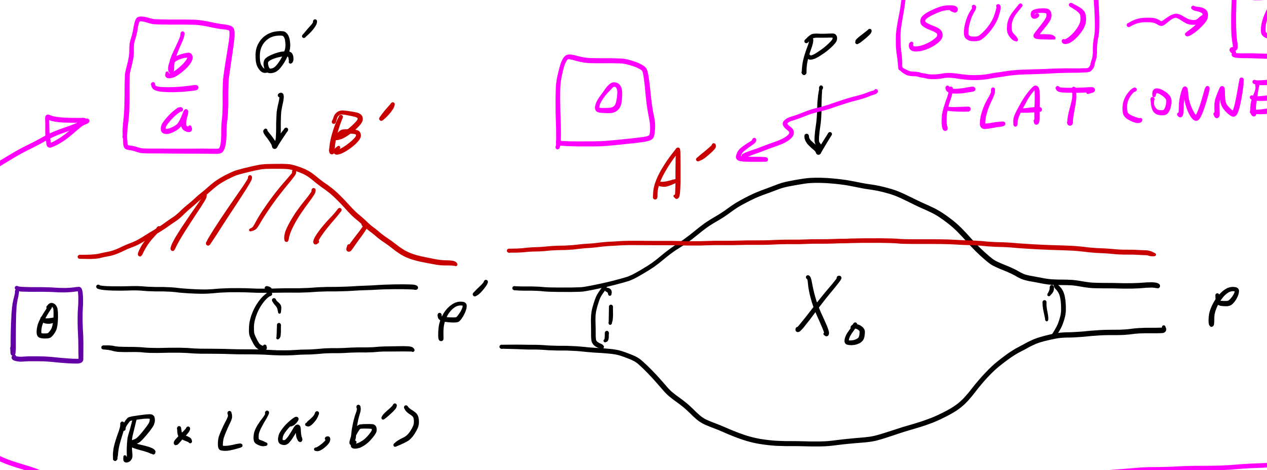
$$\frac{1}{8\pi^2}\|F_A\|_{L^2}^2 = c_2(\mathbb{P}) = \frac{b}{a}$$

A : instanton



② Sliding ends
$X_0 \times SU(2)$
$\boxed{0}$
$\theta \# B$
$\boxed{\frac{b}{a}}$
Q
Taubes' gluing of instantons
$b_2^+(X) = 0$
✻ $H_1(X;\mathbb{Z}) = 0$
θ
X_0
θ
P
$\exists$ an end
$\mathbb{R} \times L(a,b)$
$[A_n] \in \mathcal{M}(P)$
$\boxed{1} > \frac{b}{a}$
P
S^4
limit
STOP
$E = \frac{1}{8\pi^2} \|F^+_{A_n}\|^2_{L^2} = \boxed{\frac{b}{a}}$
A_n
$\mathcal{M}(P)$
limit
θ
X_0
P
$L(a',b')$
$L(a,b)$
$\dim \mathcal{M}(P) = 1$
$\boxed{?}$
Q'
B'
$\boxed{?}$
P'
SU(2) FLAT CONNECTION
A'
$\exists$ other end
θ
P'
X_0
P
$\mathbb{R} \times L(a',b')$
$\# \operatorname{Sing} \mathcal{M}(P) = \mu = 0$
✻ Under perturbed eq. ✻✻ in the next slide.

③ Type of the lens space & the Limiting flat connection



$\boxed{SU(2)} \rightsquigarrow \boxed{U(1)}$ by perturbation

FLAT CONNECTION

☆☆ $F_A^+ + \tau(A) = 0$

assumption

$$\left\{ \begin{array}{l} \boxed{a \geq a'} \\ \boxed{\dfrac{b}{a} \leq \dfrac{|b'|}{a'}} \end{array} \right.$$

$$\boxed{\begin{array}{l} \mu(Q') \ni [B'] \\ a' \gg |b'| \end{array}}$$

$\leq c_2(Q') \leq \boxed{\frac{b}{a}}$

$$\Rightarrow \boxed{\begin{array}{ll} a' = a & \\ b' = -b & \rho \cong \rho' \end{array}}$$

$L(a', b') = L(a, -b)$

$$\boxed{\pi_1(X_0) \twoheadrightarrow \mathbb{Z}_a}$$

$\rightsquigarrow$ $\mathbb{Z}_a$- covering $\widetilde{X}_0$

$\widetilde{X} := \widetilde{X_0}^{\wedge}$ (Fox compl.) $\square$

☆☆ Since the formal dim. of the moduli space of flat connections on the limiting bundle is negative.

We can perturb away irreducible flat connections except reducibles.

$\boxed{4}$ Concluding Remark

① It seems difficult to prove this result
by using SW-theory / OS-theory ...

$\boxed{\text{0-instanton = flat connection}}$

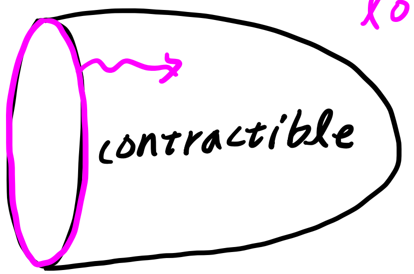
$\boxed{Q}$ What is SW-theory / OS-theory counterpart ?

② $\mathcal{M}_X(\mathbb{P}) \to \mathcal{M}_{\tilde{X}}(\tilde{\mathbb{P}})^{\mathbb{Z}_a} \subset \mathcal{M}_{\tilde{X}}(\tilde{\mathbb{P}}) \circlearrowleft \mathbb{Z}_a$

[N. Anvari - I. Hambleton (2016)]

$\exists\ \infty$-families of

$\mathbb{Z}_a \curvearrowright$



action extends
locally linearly
but
not smoothable

Σ : Seifert $\mathbb{Z}HS^3$

References

[1] N. Anvari, I. Hambleton, Cyclic group actions on contractible 4-manifolds, Geom. Topol. Volume 20, Number 2 (2016), 1127–1155.

[2] D. H. Choe, K. Park, On intersection forms of definite 4-manifolds bounded by a rational homology 3-sphere, arXiv:1704.04419v2 [math.GT] 21 Feb 2018.

[3] Y. Fukumoto, On negative-definite cobordisms among lens spaces of type (m, 1) and uniformization of four-orbifolds, preprint.

[4] M. Furuta, On self-dual pseudo-connections on some orbifolds. Math. Z. 209 (1992), no. 3, 319–337.

[5] M. Furuta, Za-invariant SU(2) instantons over the Four Sphere. Geometry of Low-Dimensional Manifolds: 1 Gauge theory and algebraic surfaces, Proceedings of the Durham Symposium, July 1989, London Mathematical Society Lecture Note Series 150, (1990) 161–174.

[6] M. Kato, On uniformization of orbifolds, Adv. Stud. in Pure. Math 9 (1986) Homotopy Theory and Related Topics, 149–172.

[7] P. Lisca, Sums of lens spaces bounding rational balls, Algebraic and Geometric Topology 7 (2007) 2141–2164.

[8] M. Namba, Branched coverings and algebraic functions, Pitman research notes in mathematics series 161, Longman Scientific & Technical, 1987.