Non-orientable Lagrangian surfaces in rational 4-manifolds and symplectic packing problems

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Today:
$X = X_k = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$: rational 4-manifold
L: non-orientable Lagrangian surface within a fixed homology class $A \in H_2(X; \mathbb{Z}_2)$

Examples

- cotangent bundle:
  L: smooth surface
  λ_{can}: canonical 1-form
  $\omega_{can} = -d\lambda$: a symplectic form on the cotangent bundle T^*L.
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- $\mathbb{C}^2$ with coordinates $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$
  $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$
  $T_{Clifford} = \{|z_1| = 1\} \times \{|z_2| = 1\}$: Clifford torus
  $\Rightarrow$ $T_{Clifford}$ is Lagrangian

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- $\mathbb{CP}^2$, with Fubini-Study form ω_{FS}.
  $L \subset \mathbb{CP}^2$: real part $\Rightarrow L$ is Lagrangian, $L \cong \mathbb{RP}^2, [L] \neq 0$

Constraints

Theorem (Weinstein)

Let (X, ω) be a symplectic manifold and $L \subset X$ a compact Lagrangian submanifold. Then there exists a neighborhood $N(L) \subset T^*L$ of the zero section, a neighborhood $V \subset X$ of L, and a diffeomorphism $\phi : N(L) \to V$ such that

$$\phi^*\omega = \omega_{can}, \phi|_L = id.$$

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For example, if L is non-orientable Lagrangian and $[L] = 0$, then $\chi(L) \equiv 0(mod 4), L \cong 2\mathbb{RP}^2 = KB, 6\mathbb{RP}^2, 10\mathbb{RP}^2, \cdots$.

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- (Shevchishin) $L \subset X_k$: Lagrangian, $[L] = 0 \Rightarrow L \ncong KB$.

Main theorem

Theorem (Dai-H.-Li)

Let X be a rational 4-manifold and $A \in H_2(X;\mathbb{Z}_2)$. Then A is represented by an embedded non-orientable Lagrangian surface or a sphere of Euler number χ for some symplectic structure if and only if

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Idea of proof:
find Lagrangian $m\mathbb{RP}^2$ with small m
+"Lagrangian" connected sum
+Lagrangian blowup

Lagrangian connected sum

- Lagrangian surgery (Polterovich):
  Let $L_1, L_2 \subset X$ be two Lagrangian surfaces intersecting transversally at one point. Then there exists a Lagrangian surface L' given by smoothing the intersection point. In particular, $L' \cong L_1 \# L_2, [L'] = [L_1] + [L_2]$.

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- Let $L \subset X$ be an immersed Lagrangian surface with a transversal self-intersection point. Then there exists an embedded Lagrangian surface L' given by smoothing the intersection point. In particular, $[L'] = [L]$ and $L' \cong L \# T^2$ or $L' \cong L \# KB$.

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- Let $L \subset X$ be an immersed Lagrangian surface with a transversal self-intersection point. Then there exists an embedded Lagrangian surface L' given by smoothing the intersection point. In particular, $[L'] = [L]$ and $L' \cong L \# T^2$ or $L' \cong L \# KB$.
- Let $L_1, L_2 \subset X$ be two disjoint Lagrangian surfaces. Then there exists a Lagrangian surface L' given by perturbing L_2 to create two intersection points and applying Lagrangian surgery. $L' \cong L_1 \# L_2 \# 2\mathbb{RP}^2, [L'] = [L_1] + [L_2]$.
  In particular, if $L_2 = T^2, [L_2] = 0$, then
  $L' \cong L_1 \# 4\mathbb{RP}^2, [L'] = [L]$.

Symplectic blowup:
Let $\tilde{B} = \{(z, l) \in \mathbb{C}^2 \times \mathbb{CP}^1 \,|\, z \in l\}$, $\tilde{B}_\varepsilon = \{(z, l) \in \tilde{B} \,|\, |z| \leq \varepsilon\}$, and $B_r = \{z \in \mathbb{C}^2 \,|\, |z| \leq r\}$. There are two natural projections $p_1 : \tilde{B} \to \mathbb{C}^2$, and $p_2 : \tilde{B} \to \mathbb{CP}^1$. p_1 implies that $\tilde{B}$ is the blowup of $\mathbb{C}^2$ at the origin.
For any $\lambda > 0$, let $\omega_\lambda = p_1^*\omega_0 + \lambda^2 p_2^*\omega_{FS}$ be the induced symplectic form on $\tilde{B}$.
There is a symplectomorphism

$$\alpha : (\tilde{B}_\varepsilon - \mathbb{CP}^1, \omega_\lambda) \cong (B_{\sqrt{\lambda^2+\varepsilon^2}} - \overline{B_\lambda}, \omega_0).$$

Let X be a symplectic 4-manifold, $x \in U \subset X$ and $\delta > \sqrt{\lambda^2 + \varepsilon^2}$, $\phi : (U, \omega) \to (B_\delta, \omega_0)$ a symplectomorphism with $\phi(x) = 0$. A symplectic blowup of X at x is $X' = (X - \phi^{-1}(B_\lambda) \cup \tilde{B}_\varepsilon / \sim$ where $a \sim b \Leftrightarrow a = \alpha(b)$ and a sympelctic form ω' on X' is induced by ω and ω_λ.
Let $p : X' \to X$ the project map, $E = PD(\mathbb{CP}^2) \in H^2(X'; \mathbb{Z})$.
Then $[\omega'] = [p^*\omega] + \pi\lambda^2 E$.

Lagrangian blowup:
(Rieser) $L \subset X, x \in L$. There exists a symplectic manifold $(\tilde{X}, \tilde{\omega})$, a Lagrangian submanifold $\tilde{L} \subset \tilde{X}$, a smooth onto map $p : \tilde{X} \to X$ such that

- $p : p^{-1}(X - x) \to X - x$ is a diffeomorphism.
- $p^{-1}(x) \cong \mathbb{CP}^1$.
- $E = PD(p^{-1}(x))$, $[\tilde{\omega}] = [p^*\omega] + \pi\lambda^2 E$.
- $p(\tilde{L}) = L, \tilde{L} \cong L\#\mathbb{RP}^2$, $[\tilde{L}] = p^*[L] + E(mod\ 2)$.

Symplectic/Lagrangian/relative blowup

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Relative blowup:
$L \subset (X, \omega)$: Lagrangian, $x \in X - L$. There exists a symplectic blowup $p : X' \to X$ at x such that $L' = p^{-1}(L)$ is Lagrangian and $L' \cong L, [L'] = p^*[L]$.

Symplectic cone

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  $C(X) = \{[\omega] | \omega \in \Omega(X)\}$: symplectic cone of X

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- $\mathcal{K}(X) = \{\text{symplectic canonical classes}\}$
  $C_K(X) = \{[\omega] | \omega \in \Omega(X), K_\omega = K\}$
  $K \in \mathcal{K}(X)$, $\mathcal{E}_K(X) = \{e \in H^2(X;\mathbb{Z}) | PD(e)$ is represented by a smoothly embedded (-1)-sphere, $e \cdot K = -1\}$.

Theorem (Li-Liu)

Let X be a closed oriented smooth 4-manifold and $b^+(X) = 1, C(X) \neq \emptyset$.

1. *Any class in $\mathcal{E}_K(X)$ is represented by a ω-symplectic (-1)-sphere for any $\omega \in C_K(X)$.*
2. $C_K(X) = \{A \in \mathcal{FP} | A \cdot e > 0, \forall e \in \mathcal{E}_K(X)\}$
3. $|\mathcal{K}(X)| < \infty$ *and $\mathcal{K}(X)$ is transitive under diffeomorphism.*

Symplectic cone

Let $\{H, E_1, \cdots, E_k\}$ be a standard basis of $H^2(X_k; \mathbb{Z})$,
$H^2 = 1, H \cdot E_i = 0, E_i, \cdots, E_j = -\delta_{ij}$.
Assume the canonical class is $K_0 = -3H + E_1 + \cdots + E_k$.
$[\omega] = aH - \sum b_i E_i \in C_K(X)$

$$\Leftrightarrow \begin{cases} a^2 - \sum b_i^2 > 0 \\ a > 0, b_i > 0 \\ e = pH - \sum q_i E_i \in \mathcal{E}_K(X), [\omega] \cdot e = ap - \sum b_i q_i > 0 \end{cases}$$

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Examples: Consider $NC(X) = \{aH - \sum b_i E_i \in C(X) | a = 1\}$

- In X_2, $E_1, E_2, H - E_1 - E_2 \in \mathcal{E}_K(X_2)$
$$\begin{cases} b_1^2 + b_2^2 < 1 \\ b_1, b_2 > 0 \\ b_1 + b_2 < 1 \end{cases}$$
- In X_3, $E_1, E_2, E_3, H - E_i - E_j \in \mathcal{E}_K(X_2)$
$$\begin{cases} b_1^2 + b_2^2 + b_3^2 < 1 \\ b_1, b_2, b_3 > 0 \\ b_i + b_j < 1 \end{cases}$$

Relative symplectic cone

- $n \in \mathbb{N}, A \in H_2(X;\mathbb{Z}_2)$,
  $C_{n,A}(X) = \{[\omega] | \omega \in \Omega(X), \exists L$: nonorientable ω-Lagrangian surface, $[L] = A, L \cong n\mathbb{RP}^2\}$.
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- Question:
  1. What is $C_{n,A}(X)$? What is $C_n(X)$?
  2. $C_{n,A}(X) = C_{n+4,A}(X)$?

The structure of symplectic cone is related to the symplectic packing problem.

Symplectic packing

- A symplectic packing of (X, ω) is a symplectic embedding

$$\varphi(c_1, \cdots, c_k) : \coprod B(\sqrt{\frac{c_i}{\pi}}, \omega_0) \to (X, \omega)$$

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- Applying symplectic blowup,
$\varphi(c_1, \cdots, c_k) \in Emb(X, \omega) \Longrightarrow [p^*\omega] - \sum c_i E_i \in C(X \# \overline{\mathbb{CP}^2})$

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- (McDuff-Polterovich) $X = X_k, k \leq 8$,
$H - \sum c_i E_i \in NC_{K_0}(X) \Rightarrow \exists \varphi(c_1, \cdots, c_k) \in Emb(\mathbb{CP}^2, \omega_{FS})$.

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$H - \sum c_i E_i \in NC_{K_0}(X) \Rightarrow \exists \varphi(c_1, \cdots, c_k) \in Emb(\mathbb{CP}^2, \omega_{FS})$.
- Let Z be a submanifold of (X, ω). A relative (symplectic) packing of (X, Z) is a symplectic packing
$$\varphi(c_1, \cdots, c_k) : \coprod B(\sqrt{\frac{c_i}{\pi}}, \omega_0) \to (X - Z, \omega)$$
i.e. $\varphi(c_1, \cdots, c_k) \in Emb(X - Z, \omega)$
Similarly, a relative packing $\varphi(c_1, \cdots, c_k) \in Emb(X - Z, \omega)$ induces a symplectic structure in $X \# \overline{\mathbb{CP}^2}$. But we don't know if the converse is true.

- (Borman-Li-Wu) Symplectic packing for $(\mathbb{CP}^2 - \mathbb{RP}^2, \omega_{FS})$ and $(S^2 \times S^2, \Omega_{1,\frac{1}{2}})$ are equivalent.

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- We can view $S^2 \times S^2 \sharp k \overline{\mathbb{CP}^2}$ as $X' = \mathbb{CP}^2 \sharp (k+1) \overline{\mathbb{CP}^2}$ with standard basis $H', E_1', \cdots, E_{k+1}'$.
$\omega = H - c_1 E_1 - \cdots c_k E_k \in C_{1,H}(X_k)$ corresponds to a symplectic form

$$\omega' = (\frac{3}{2} - c_1)H' - (1 - c_1)E_1' - (\frac{1}{2} - c_1)E_2' - \sum_{i=2}^{k} c_i E_{i+1}' \in C(X_{k+1})$$

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- Use Cremona transformation, it is enough to understand $C_{1,H}(X_k)$.

$C_1(X_k)$

Observations:

- $C_{1,H}(X_k)$ can be embedded to $C(X_{k+1})$ as a hyper surface.
- $C_{1,H}(X_k) \subset C(X_k)$ is bounded by some hyperplane with symmetric coefficients.
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Questions:

- Is $C_{1,H}(X_k)$ a polyhedron?
- Can $C_{1,H}(X_8)$ be determined by finite many hyperplanes in $C(X_9)$?
- More symmetric structures on $C_{1,H}(X_k)$, $C_1(X_k)$.

Thank you