Pin(2)-equivariant maps between vector bundles over tori and KO-degree

Ko Ohashi

The University of Tokyo

September 12, 2018

This talk is based on joint work with M. Furuta and Y. Kametani.

Table of Contents

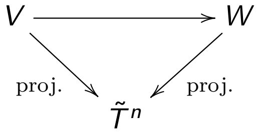
Introduction

- This talk concerns Pin(2)-equivariant maps between Pin(2)-vector bundles over tori.

$$\begin{array}{ccc} V & \longrightarrow & W \\ & {\scriptstyle \text{proj.}} \searrow \quad \swarrow {\scriptstyle \text{proj.}} & \\ & \tilde{T}^n & \end{array}$$

Introduction

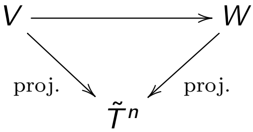
- This talk concerns Pin(2)-equivariant maps between Pin(2)-vector bundles over tori.



- I will explain the background and motivation of this talk.

Introduction

- This talk concerns Pin(2)-equivariant maps between Pin(2)-vector bundles over tori.



- I will explain the background and motivation of this talk.

Convention

- All manifolds are assumed to be connected, closed, and smooth.
- Pin(2) is defined as the subset $S^1 \cup jS^1$ of the quaternions $\mathbb{H}$.

The Monopole Map

- Let X be a spin 4-manifold with indefinite intersection form and $b_2^+(X) > 0$. Fix a Riemannian metric on X.

The Monopole Map

- Let X be a spin 4-manifold with indefinite intersection form and $b_2^+(X) > 0$. Fix a Riemannian metric on X.
- We consider the monopole equation as the map

$$\Phi : \mathrm{Ker}(\Omega^1 \xrightarrow{d^*} \Omega^0) \oplus \Gamma(S^+) \to H^1(X;\mathbb{R}) \times (\Omega^+ \oplus \Gamma(S^-)),$$
$$\Phi(a,s) = ((a)_{\text{harmonic}}, d^+a + q(s), D_a s)$$

where

- Ω^p is the set of all differential p-forms on X,
- $\Gamma(S^\pm)$ is the set of all sections of the half-spinor bundles $S^\pm$ of X,
- $q : \Gamma(S^+) \to \Omega^+$ is the quadratic map,
- $(a)_{\mathrm{harmonic}}$ is the harmonic part of $a \in \Omega^1$,
- D is the Dirac operator.

Symmetry

The monopole map

$$\Phi : \mathrm{Ker}(\Omega^1 \xrightarrow{d^*} \Omega^0) \oplus \Gamma(S^+) \to H^1(X;\mathbb{R}) \times (\Omega^+ \oplus \Gamma(S^-))$$

Symmetry

The monopole map

$$\Phi : \mathrm{Ker}(\Omega^1 \xrightarrow{d^*} \Omega^0) \oplus \Gamma(S^+) \to H^1(X;\mathbb{R}) \times (\Omega^+ \oplus \Gamma(S^-))$$

- The spin structure of X provides the Pin(2)-equivariance of Φ.
  - Pin(2) acts on Ω^p and $H^1(X;\mathbb{R})$ via $\mathrm{Pin}(2) \to \mathrm{Pin}(2)/S^1 = \{\pm 1\}$.
  - Pin(2) acts on $\Gamma(S^\pm) = \Gamma(P \times_{\mathrm{Spin}} \mathbb{H}^\pm)$ via $\mathrm{Pin}(2) \to \mathrm{Sp}(1)$.
  - Φ is $\mathrm{Pin}(2)$-equivariant.

Symmetry

The monopole map

$$\Phi : \mathrm{Ker}(\Omega^1 \xrightarrow{d^*} \Omega^0) \oplus \Gamma(S^+) \to H^1(X;\mathbb{R}) \times (\Omega^+ \oplus \Gamma(S^-))$$

- The spin structure of X provides the Pin(2)-equivariance of Φ.
  - Pin(2) acts on Ω^p and $H^1(X;\mathbb{R})$ via $\mathrm{Pin}(2) \to \mathrm{Pin}(2)/S^1 = \{\pm 1\}$.
  - Pin(2) acts on $\Gamma(S^\pm) = \Gamma(P \times_{\mathrm{Spin}} \mathbb{H}^\pm)$ via $\mathrm{Pin}(2) \to \mathrm{Sp}(1)$.
  - Φ is $\mathrm{Pin}(2)$-equivariant.
- The gauge symmetry provides the $H^1(X;\mathbb{Z})$-equivariance of Φ.
  - $a \in H^1(X;\mathbb{Z})$ acts on $\omega \in \Omega^1, \eta \in \Omega^+$ and $s \in \Gamma(S^\pm)$ by

    $$a \cdot \omega = (a)_{harmonic} + \omega,\ \ a \cdot \eta = \eta,\ \ a \cdot s = \exp(2\pi\sqrt{-1}\rho(a))s.$$

    Here $\rho : H^1(X;\mathbb{Z}) \to C^\infty(X, \mathbb{R}/\mathbb{Z})$ is defined as

    $$\rho([\omega])(x) = \int_{x_0}^{x} \omega \mod \mathbb{Z}, \quad x \in X,\ \omega \in \Omega^1.$$

  - Φ is $H^1(X;\mathbb{Z})$-equivariant.

Finite-dimensional approximation

- By taking the quotient by $H^1(X;\mathbb{Z})$ and a finite dimensional approximation of the map

$$\Phi : \mathrm{Ker}(\Omega^1 \xrightarrow{d^*} \Omega^0) \oplus \Gamma(S^+) \to H^1(X;\mathbb{R}) \times (\Omega^+ \oplus \Gamma(S^-)),$$

we obtain

$$\begin{array}{ccc} \phi : V & \longrightarrow & W \\ & \searrow^{\text{proj.}} \quad {}^{\text{proj.}}\swarrow & \\ & J_X & \end{array},$$

where V and W are finite dimensional real Pin(2)-vector bundles over the Jacobian torus $J_X = H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$.

Properties of a finite-dimensional approximation ϕ

1. $\phi : V \to W$ is a proper, fiber-preserving, Pin(2)-equivariant map.

Properties of a finite-dimensional approximation ϕ

1. $\phi : V \to W$ is a proper, fiber-preserving, Pin(2)-equivariant map.
2. The linear action of $-1 \in \mathrm{Pin}(2)$ on V induces the decomposition $V = V_0 \oplus V_1$. Here $V_0 = \{v \mid (-1) \cdot v = v\}$ and $V_1 = \{v \in V \mid (-1) \cdot v = -v\}$. Similarly we have $W = W_0 \oplus W_1$.

Properties of a finite-dimensional approximation ϕ

1. $\phi : V \to W$ is a proper, fiber-preserving, Pin(2)-equivariant map.
2. The linear action of $-1 \in \mathrm{Pin}(2)$ on V induces the decomposition $V = V_0 \oplus V_1$. Here $V_0 = \{v \mid (-1) \cdot v = v\}$ and $V_1 = \{v \in V \mid (-1) \cdot v = -v\}$. Similarly we have $W = W_0 \oplus W_1$.
3. $\phi|_{V_0} : V_0 \to W_0$ satisfies the commutative diagram

$$\begin{array}{ccc} V_0 & \xrightarrow{\phi} & W_0 \\ \cong\uparrow & & \uparrow\cong \\ J_X \times \tilde{\mathbb{R}}^x & \xrightarrow[\text{inclusion}]{} & J_X \times \tilde{\mathbb{R}}^{x+\ell} \end{array}$$

where $\tilde{\mathbb{R}}$ is the 1-dimensional real representation of Pin(2) defined by $j \cdot x = -x$ and $z \cdot x = x$ for $z \in S^1, x \in \tilde{\mathbb{R}}$.

Properties of a finite-dimensional approximation ϕ

1. $\phi : V \to W$ is a proper, fiber-preserving, Pin(2)-equivariant map.
2. The linear action of $-1 \in \mathrm{Pin}(2)$ on V induces the decomposition $V = V_0 \oplus V_1$. Here $V_0 = \{v \mid (-1) \cdot v = v\}$ and $V_1 = \{v \in V \mid (-1) \cdot v = -v\}$. Similarly we have $W = W_0 \oplus W_1$.
3. $\phi|_{V_0} : V_0 \to W_0$ satisfies the commutative diagram

$$\begin{array}{ccc} V_0 & \xrightarrow{\ \phi\ } & W_0 \\ \cong \uparrow & & \uparrow \cong \\ J_X \times \tilde{\mathbb{R}}^x & \xrightarrow[\text{inclusion}]{} & J_X \times \tilde{\mathbb{R}}^{x+\ell} \end{array}$$

   where $\tilde{\mathbb{R}}$ is the 1-dimensional real representation of Pin(2) defined by $j \cdot x = -x$ and $z \cdot x = x$ for $z \in S^1, x \in \tilde{\mathbb{R}}$.
4. V_1 and W_1 are the realification of "sp-bundles" V_{sp} and W_{sp}. such that some class of the "Ksp-invariants" of the difference between W_{sp} and V_{sp} vanish.

Main results

1. A Borsuk-Ulam type inequality for the existence of Pin(2)-equivariant maps
2. A 10/8-type inequality for $b_1(X) \geq 0$
3. Determination of the KO_Γ-degree of ϕ for $b_2^+(X)$ even

Contents

sp-bundle

- Let B be a C_2-space (C_2 denotes the cyclic group of order two).
- $j_B : B \to B$: the involution on B.

Definition (Dupont)

A sp-bundle is a complex vector bundle V over B with an antilinear map $J : V \to V$ satisfying the following conditions.

1. *J is a lift of j_B. That is $\pi_V \circ J = j_B \circ \pi_V$.*
2. $J \circ J = -\mathrm{id}_V$.

Ksp-group

Definition

For any compact Hausdorff C_2-space B, the Ksp-group $\mathrm{Ksp}(B)$ *of B is defined by*

$$\mathrm{Ksp}(B) = F(B)/Q(B),$$

- $F(B)$ *is the free abelian group generated by the isom. classes of sp-bundles over B.*
- $Q(B)$ *is the subgroup of* $F(B)$ *generated by* $[V] + [W] - [V \oplus W],\ V, W$: *sp-bundles*

Ksp-group

Definition

For any compact Hausdorff C_2-space B, the Ksp-group $\mathrm{Ksp}(B)$ *of B is defined by*

$$\mathrm{Ksp}(B) = F(B)/Q(B),$$

- *$F(B)$ is the free abelian group generated by the isom. classes of sp-bundles over B.*
- *$Q(B)$ is the subgroup of $F(B)$ generated by* $[V] + [W] - [V \oplus W],\ V, W$: *sp-bundles*

Definition

For any locally compact C_2-space U, $\mathrm{Ksp}(U)$ *is defined by*

$$\mathrm{Ksp}(U) = \mathrm{Ker}(\mathrm{Ksp}(U^+) \to \mathrm{Ksp}(\{\infty\})),$$

where $U^+ = U \cup \{\infty\}$ is the one-point compactification of U.

Computations

Example

1. The Ksp-group of $\tilde{\mathbb{R}}^p$ (with $j_B(v) = -v$) is given by

$$\mathrm{Ksp}(\tilde{\mathbb{R}}^p) \cong \begin{cases} \mathbb{Z} & p \equiv 0, 4 \bmod 8 \\ \mathbb{Z}/2\mathbb{Z} & p \equiv 2, 3 \bmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Computations

Example

1. The Ksp-group of $\tilde{\mathbb{R}}^p$ (with $j_B(v) = -v$) is given by

$$\mathrm{Ksp}(\tilde{\mathbb{R}}^p) \cong \begin{cases} \mathbb{Z} & p \equiv 0, 4 \bmod 8 \\ \mathbb{Z}/2\mathbb{Z} & p \equiv 2, 3 \bmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $\tilde{T}^n$ be the n-dimensional torus $\tilde{\mathbb{R}}^n/\mathbb{Z}^n$. The Ksp-group of $\tilde{T}^n$ can be computed by

$$\mathrm{Ksp}(\tilde{T}^n) \cong \bigoplus_{S \subset [n]} \mathrm{Ksp}(\tilde{\mathbb{R}}^S),$$

where the symbol $[n]$ stands for the set $\{1, 2, \ldots, n\}$ and $\tilde{\mathbb{R}}^S$ denotes the set of all maps from S to $\tilde{\mathbb{R}}$.

The realification of a sp-bundle

Remark

For any C_2-space B, we regard B as a $\mathrm{Pin}(2)$-space via the map $\mathrm{Pin}(2) \to \mathrm{Pin}(2)/S^1 = C_2$.

Definition

The realification of a sp-bundle (V, J) is the underlying real vector bundle rV with the Pin(2)-action defined by

$$z \cdot v := zv, \quad j \cdot v := J(v)$$

for $v \in rV, z \in S^1$.

sp-bundles $\rightsquigarrow$ real Pin(2)-vector bundles

Contents

Setting

- Let V and W be Pin(2)-vector bundles over $\tilde{T}^n$, $n \geq 0$.

Setting

- Let V and W be Pin(2)-vector bundles over $\tilde{T}^n$, $n \geq 0$.
- Given a proper, fiber-preserving, Pin(2)-map $\phi : V \to W$.

Setting

- Let V and W be Pin(2)-vector bundles over $\tilde{T}^n$, $n \geq 0$.
- Given a proper, fiber-preserving, Pin(2)-map $\phi : V \to W$.
- The linear action by $-1 \in \mathrm{Pin}(2)$ on V, W defines the decompositions $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, respectively.

Setting

- Let V and W be Pin(2)-vector bundles over $\tilde{T}^n$, $n \geq 0$.
- Given a proper, fiber-preserving, Pin(2)-map $\phi : V \to W$.
- The linear action by $-1 \in \mathrm{Pin}(2)$ on V, W defines the decompositions $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, respectively.

$$\begin{array}{ccc} V = V_0 \oplus V_1 & \xrightarrow{\quad \phi \quad} & W = W_0 \oplus W_1 \\ & {\scriptstyle \text{proj.}} \searrow \quad \swarrow {\scriptstyle \text{proj.}} & \\ & \tilde{T}^n & \end{array}.$$

Setting

- Let V and W be Pin(2)-vector bundles over $\tilde{T}^n$, $n \geq 0$.
- Given a proper, fiber-preserving, Pin(2)-map $\phi : V \to W$.
- The linear action by $-1 \in \mathrm{Pin}(2)$ on V, W defines the decompositions $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, respectively.

$$\begin{array}{ccc} V = V_0 \oplus V_1 & \xrightarrow{\quad \phi \quad} & W = W_0 \oplus W_1 \, . \\ & {\scriptstyle \text{proj.}} \searrow \qquad \swarrow {\scriptstyle \text{proj.}} & \\ & \tilde{T}^n & \end{array}$$

Assumptions

- V_0, W_0 are given by $V_0 = \tilde{T}^n \times \tilde{\mathbb{R}}^x$, $W_0 = \tilde{T}^n \times \tilde{\mathbb{R}}^{x+\ell}$ with $\ell > 0$.
- $\phi|_{V_0} : V_0 \to W_0$ is given by the inclusion map $\tilde{\mathbb{R}}^x \hookrightarrow \tilde{\mathbb{R}}^{x+\ell}$.
- V_1, W_1 are given by the realifications of sp-bundles V_{sp}, W_{sp} over $\tilde{T}^n$, respectively.

Ksp-invariant

The sp-bundles V_{sp}, W_{sp} define an element $a := [W_{sp}] - [V_{sp}] \in \mathrm{Ksp}(\tilde{T}^n)$. Using the isomorphism $\mathrm{Ksp}(\tilde{T}^n) \cong \bigoplus_{S\subset[n]} \mathrm{Ksp}(\tilde{\mathbb{R}}^S)$, it can be decomposed as

$$a = \sum_{S\subset[n]} a_S$$

where

$$a_S \in \mathrm{Ksp}(\tilde{\mathbb{R}}^S) \cong \begin{cases} \mathbb{Z} & |S| \equiv_8 0, 4 \\ \mathbb{Z}/2\mathbb{Z} & |S| \equiv_8 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Ksp-invariant

The sp-bundles V_{sp}, W_{sp} define an element $a := [W_{sp}] - [V_{sp}] \in \mathrm{Ksp}(\tilde{T}^n)$. Using the isomorphism $\mathrm{Ksp}(\tilde{T}^n) \cong \bigoplus_{S \subset [n]} \mathrm{Ksp}(\tilde{\mathbb{R}}^S)$, it can be decomposed as

$$a = \sum_{S \subset [n]} a_S$$

where

$$a_S \in \mathrm{Ksp}(\tilde{\mathbb{R}}^S) \cong \begin{cases} \mathbb{Z} & |S| \equiv_8 0, 4 \\ \mathbb{Z}/2\mathbb{Z} & |S| \equiv_8 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Assumption

If $|S| > 4$, then $a_S = 0$.

Invariant ν_S

Invariant ν_S

1. Let S be a subset of $[n] = \{1, 2, \ldots, n\}$.

Invariant ν_S

1. Let S be a subset of $[n] = \{1, 2, \ldots, n\}$.
2. The integer $N_m(S), m \geq 0$ is defined to be the number of families $\{S_1, \ldots, S_m\}$ of S such that

$$S_i \neq S_j \ (i \neq j), \ S = S_1 \cup \cdots \cup S_m, \ |S_i| \in \{2, 3, 4\},$$
$$a_{S_i} \neq 0 \in \mathrm{Ksp}(\tilde{\mathbb{R}}^{S_i}) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

Invariant ν_S

1. Let S be a subset of $[n] = \{1, 2, \ldots, n\}$.
2. The integer $N_m(S), m \geq 0$ is defined to be the number of families $\{S_1, \ldots, S_m\}$ of S such that

$$S_i \neq S_j \ (i \neq j), \ S = S_1 \cup \cdots \cup S_m, \ |S_i| \in \{2, 3, 4\},$$
$$a_{S_i} \neq 0 \in \mathrm{Ksp}(\tilde{\mathbb{R}}^{S_i}) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

3. The integer $N(S)$ is defined by

$$N(S) = \sum_{m=0}^{\infty} (-2)^m N_m(S).$$

Invariant ν_S

1. Let S be a subset of $[n] = \{1, 2, \dots, n\}$.
2. The integer $N_m(S), m \geq 0$ is defined to be the number of families $\{S_1, \dots, S_m\}$ of S such that

$$S_i \neq S_j \ (i \neq j), \ S = S_1 \cup \cdots \cup S_m, \ |S_i| \in \{2, 3, 4\},$$
$$a_{S_i} \neq 0 \in \mathrm{Ksp}(\tilde{\mathbb{R}}^{S_i}) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

3. The integer $N(S)$ is defined by

$$N(S) = \sum_{m=0}^{\infty} (-2)^m N_m(S).$$

4. The integer ν_S is defined by

$$\nu_S := \max\{\nu \mid 2^\nu \text{ divides } N(S)\}$$

Main result

Theorem

If there exists a Pin(2)-map $\phi : V = V_0 \oplus V_1 \to W = W_0 \oplus W_1$ *satisfying the assumptions, then we have the inequality*

$$\ell_S \geq 2k_S + \varepsilon(k_S, \ell, |S|)$$
$$(\Leftrightarrow \ \ell \geq 2k + |S| - 2\nu_S + \varepsilon(k - \nu_S, \ell, |S|)).$$

Here $\ell_S = \ell - |S|, k_S = k - \nu_S$, *and* $\varepsilon(k_S, \ell, |S|)$ *is defied by the following table.*

	$\lvert S\rvert$: even				$\lvert S\rvert$: odd			
p	0	1	2	3	0	1	2	3
$\varepsilon(p, \ell, \lvert S\rvert)$	3 ($l \neq 2$) 1 ($l = 2$)	1	2	3	2	1	2	2

Contents

Setting

- Let X be a spin 4-manifold with indefinite intersection form and $b_2^+(X) > 0$.
- Let $\phi : V \to W$ be a finite-dimensional approximation of the monopole map Φ.
- $\ell := b_2^+(X), \quad k := -\operatorname{sign}(X)/16$.
- Let $a(X) = \sum_{S \subset [n]} a_S(X)$ be the Ksp-invariant.

Definition of Σ_L

- Let L be a sublattice of $H^1(X;\mathbb{Z})$ with rank r.

Definition of Σ_L

- Let L be a sublattice of $H^1(X;\mathbb{Z})$ with rank r.
- We define the submanifold Σ_L of X as follows:

Definition of Σ_L

- Let L be a sublattice of $H^1(X;\mathbb{Z})$ with rank r.
- We define the submanifold Σ_L of X as follows:
- $T_L := \mathrm{Hom}(H^1(X;\mathbb{Z})/L, \mathbb{R}/\mathbb{Z})$ is a subtorus of $\mathrm{Hom}(H^1(X;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ of codimension r.

Definition of Σ_L

- Let L be a sublattice of $H^1(X;\mathbb{Z})$ with rank r.
- We define the submanifold Σ_L of X as follows:
- $T_L := \mathrm{Hom}(H^1(X;\mathbb{Z})/L, \mathbb{R}/\mathbb{Z})$ is a subtorus of $\mathrm{Hom}(H^1(X;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ of codimension r.
- Then $\Sigma_L := \rho^{-1}(T_L)$ is a submanifold of X of codimension r. Here $\rho : X \to \mathrm{Hom}(H^1(X;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ is the Albanese map. The spin structure of X induces a spin structure of Σ_L.

$$
\begin{array}{ccc}
T_L & \hookrightarrow & \mathrm{Hom}(H^1(X;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \\
\uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \rho} \\
\Sigma_L & \hookrightarrow & X
\end{array}
$$

Calculation of $a_S(X)$

- Fix a $\mathbb{Z}$-basis $\{x_1, \ldots, x_n\}$ of $H^1(X; \mathbb{Z})$.
- For any subset $S \subset [n]$, we define $L(S)$ as the sublattice generated by the set $\{x_s | s \in S\}$ and set $\Sigma_S := \Sigma_{L(S)}$.

Calculation of $a_S(X)$

- Fix a $\mathbb{Z}$-basis $\{x_1, \ldots, x_n\}$ of $H^1(X; \mathbb{Z})$.
- For any subset $S \subset [n]$, we define $L(S)$ as the sublattice generated by the set $\{x_s | s \in S\}$ and set $\Sigma_S := \Sigma_{L(S)}$.

Theorem

Under the isomorphism $\mathrm{Ksp}(\tilde{\mathbb{R}}^S) \cong \mathrm{KO}^{|S|-4}(\mathrm{pt.})$, *we have*

$$a_S(X) = \alpha([\Sigma_S]),$$

where α *denotes the* α*-invariant* $\alpha : \Omega_*^{spin} \to \mathrm{KO}^{-*}(\mathrm{pt.})$.

Corollary

By Theorem $a_S(X) = \alpha([\Sigma_S])$, we obtain the following corollary.

Corollary

- *If* $|S| > 4$, *then* $a_S(X) = 0$.
- *If* $|S| = 0$ *or* 4, *then it can be written (up to sign) as*

$$a_S(X) = \begin{cases} \mathrm{sign}(X)/16 & |S| = 0 \\ \int_X x_S & |S| = 4,\ x_S = \cup_{s \in S} x_s \end{cases}$$

Theorem

$$b_2^+(X) \geq -\frac{\mathrm{sign}(X)}{8} + |S| - \nu_S + \varepsilon(k_S, \ell, |S|)$$

Theorem

$$b_2^+(X) \geq -\frac{\mathrm{sign}(X)}{8} + |S| - \nu_S + \varepsilon(k_S, \ell, |S|)$$

Corollary ($S = \emptyset$)

If $b_2^+(X) \neq 2$, *then*

$$b_2^+(X) \geq -\frac{\mathrm{sign}(X)}{8} + \begin{cases} 3 & k \equiv 0, 3 \mod 4 \\ 1 & k \equiv 1 \mod 4 \\ 2 & k \equiv 2 \mod 4. \end{cases} \qquad (*)$$

Corollary ($S = \emptyset$)

If $b_2^+(X) \neq 2$, then

$$b_2^+(X) \geq -\frac{\mathrm{sign}(X)}{8} + \begin{cases} 3 & k \equiv 0, 3 \mod 4 \\ 1 & k \equiv 1 \mod 4 \\ 2 & k \equiv 2 \mod 4. \end{cases} \qquad (*)$$

Remark

1. *In the case* $k \equiv 1 \bmod 4$, *the inequality* $(*)$ *is Furuta's 10/8-inequality (2001).*
2. *In the case* $k \equiv 2, 3 \bmod 4$, *the inequality* $(*)$ *was first proved by N. Minami and B. Schimidt (2003) independently . They destabilize a Pin(2)-map and apply a result by S. Stolz on* $\mathbb{Z}/4\mathbb{Z}$*-equivariant maps.*
3. *In the case* $k \equiv 0 \bmod 4$, *J. Lin (2015) proved the inequality* $(*)$ *using the* $\mathrm{KO}_{\mathrm{Pin}(2)}$*-theoretical Euler classes, degree and the Adams operations.*

Example

Theorem$^{\text{bis}}$

$$\ell_S \geq 2k_S + \varepsilon(k_S, \ell, |S|).$$

Example

If X is decomposed as $X = X' \#(\#^m T^4)$, then we have

$$b_2^+(X) \geq -\frac{\operatorname{sign}(X)}{8} + 2m + \begin{cases} 3 & \frac{\operatorname{sign}(X)}{8} + m \equiv 0, 1 \mod 4 \\ 2 & \frac{\operatorname{sign}(X)}{8} + m \equiv 2 \mod 4. \\ 1 & \frac{\operatorname{sign}(X)}{8} + m \equiv 3 \mod 4 \end{cases}$$

This follows from Theorem by taking S as a basis of $H^1(\#^m T^4; \mathbb{Z})$. In this case, $|S| = 4m$ and $\nu_S = m$.

Contents

Thom isomorphism theorem

Notation

- B: a compact Hausdorff G-space
- V: a vector bundle V over B

$$\mathrm{KO}_G^*(V) := \widetilde{\mathrm{KO}}_G(V \cup \{\infty\}).$$

Theorem (Atiyah)

Let V be a real spin G-vector bundle over a compact Hausdorff G-space B. Then there exists $\mathrm{t}_{\mathrm{KO}}(V) \in \mathrm{KO}_G^{\dim_{\mathbb{R}} V}(V)$ such that the map

$$\mathrm{KO}_G^*(B) \to \mathrm{KO}_G^{\dim_{\mathbb{R}} V}(V),\ x \mapsto \pi^* x \cup \mathrm{t}_{\mathrm{KO}}(V),$$

is an isormophism, where $\pi : V \to B$ is the projection.

KO-Euler class and KO-degree

Definition

Let V, W be spin G-vector bundles over B and let $\phi : V \to W$ be a fibre-preserving proper G-map.

1. *The* KO_G-**Euler class** $\mathrm{e}_{\mathrm{KO}}(V) \in \mathrm{KO}_G^{\dim_{\mathbb{R}} V}(B)$ *is defined by*

$$\mathrm{e}_{\mathrm{KO}}(V) = s^*(\mathrm{t}_{\mathrm{KO}}(V))$$

   where $s : B \to V$ is the zero section.

2. *The* KO_G-**degree** $\deg_{\mathrm{KO}}(\phi) \in \mathrm{KO}_G^{\dim_{\mathbb{R}} W - \dim_{\mathbb{R}} V}(B)$ *is defined by*

$$\deg_{\mathrm{KO}}(\phi)\, \mathrm{t}_{\mathrm{KO}}(V) = \phi^*(\mathrm{t}_{\mathrm{KO}}(W)).$$

Spin structure on $\tilde{\mathbb{R}}^p$

- To define the KO-degree of ϕ, it is necessary to put Pin(2)-spin structures on $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$.

Spin structure on $\tilde{\mathbb{R}}^p$

- To define the KO-degree of ϕ, it is necessary to put Pin(2)-spin structures on $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$.
- The sp-structures on V_1, W_1 induce the Pin(2)-spin structures.

Spin structure on $\tilde{\mathbb{R}}^p$

- To define the KO-degree of ϕ, it is necessary to put Pin(2)-spin structures on $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$.
- The sp-structures on V_1, W_1 induce the Pin(2)-spin structures.
- Next we consider $V_0 \cong \underline{\tilde{\mathbb{R}}}^x$ and $W_0 \cong \underline{\tilde{\mathbb{R}}}^{x+\ell}$.
- When does $\tilde{\mathbb{R}}^p$ admit a Pin(2)-spin structure?

Spin structure on $\tilde{\mathbb{R}}^p$

- To define the KO-degree of ϕ, it is necessary to put Pin(2)-spin structures on $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$.
- The sp-structures on V_1, W_1 induce the Pin(2)-spin structures.
- Next we consider $V_0 \cong \underline{\tilde{\mathbb{R}}}^x$ and $W_0 \cong \underline{\tilde{\mathbb{R}}}^{x+\ell}$.
- When does $\tilde{\mathbb{R}}^p$ admit a Pin(2)-spin structure?

Lemma

- *The Pin(2)-representation $\tilde{\mathbb{R}}^p$ admit a Pin(2)-spin structure if and only if p is a multiple of 4.*

Spin structure on $\tilde{\mathbb{R}}^p$

- To define the KO-degree of ϕ, it is necessary to put Pin(2)-spin structures on $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$.
- The sp-structures on V_1, W_1 induce the Pin(2)-spin structures.
- Next we consider $V_0 \cong \underline{\tilde{\mathbb{R}}}^x$ and $W_0 \cong \underline{\tilde{\mathbb{R}}}^{x+\ell}$.
- When does $\tilde{\mathbb{R}}^p$ admit a Pin(2)-spin structure?

Lemma

- *The Pin(2)-representation $\tilde{\mathbb{R}}^p$ admit a Pin(2)-spin structure if and only if p is a multiple of 4.*
- *The Pin(2)-representation $\tilde{\mathbb{R}}^p$ admit a Pin(2)-invariant orientation if and only if p is even.*

The double covering Γ of $\mathrm{Pin}(2)$

Definition

We define the double covering Γ of $\mathrm{Pin}(2)$ as follows:

- *Let C_4 be the cyclic group of order 4 with a generator j_4.*
- *The C_4-action $j_4 \cdot t = t^{-1},\ t \in S^1$ defined by defines the semi direct product $S^1 \rtimes C_4$, which we denote by Γ.*
- *The double covering map $\Gamma = S^1 \rtimes C_4 \to \mathrm{Pin}(2) = S^1 \cup jS^1$ is defined by*

$$z \in S^1 \mapsto z \in S^1, \quad j_4 \in C_4 \mapsto j.$$

The double covering Γ of $\mathrm{Pin}(2)$

Definition

We define the double covering Γ of $\mathrm{Pin}(2)$ as follows:

- *Let C_4 be the cyclic group of order 4 with a generator j_4.*
- *The C_4-action $j_4 \cdot t = t^{-1},\ t \in S^1$ defined by defines the semi direct product $S^1 \rtimes C_4$, which we denote by Γ.*
- *The double covering map $\Gamma = S^1 \rtimes C_4 \to \mathrm{Pin}(2) = S^1 \cup jS^1$ is defined by*

$$z \in S^1 \mapsto z \in S^1, \quad j_4 \in C_4 \mapsto j.$$

Lemma

The Γ-representation $\tilde{\mathbb{R}}^p$ admit a Γ-spin structure if and only if p is even.

The KO-degree

Corollary

Suppose that ℓ is even.

- By stabilizing by $\tilde{\mathbb{R}}$ if necessary, $V_0 \cong \underline{\tilde{\mathbb{R}}}^x$ and $W_0 \cong \underline{\tilde{\mathbb{R}}}^{x+\ell}$ admit Γ-spin structures, and
- the KO-degree of ϕ is defined.

The KO-degree

Corollary

Suppose that ℓ is even.

- By stabilizing by $\tilde{\mathbb{R}}$ if necessary, $V_0 \cong \underline{\tilde{\mathbb{R}}}^x$ and $W_0 \cong \underline{\tilde{\mathbb{R}}}^{x+\ell}$ admit Γ-spin structures, and
- the KO-degree of ϕ is defined.

- We calculate the KO-degree $\deg_{\mathrm{KO}}(\phi)$ of ϕ in the algebra

$$\mathrm{KO}_\Gamma^*(\tilde{T}^n)[e_{\mathrm{KO}}(\mathbb{H})^{-1}].$$

The KO-degree

Corollary

Suppose that ℓ is even.

- By stabilizing by $\tilde{\mathbb{R}}$ if necessary, $V_0 \cong \underline{\tilde{\mathbb{R}}}^x$ and $W_0 \cong \underline{\tilde{\mathbb{R}}}^{x+\ell}$ admit Γ-spin structures, and
- the KO-degree of ϕ is defined.

- We calculate the KO-degree $\deg_{\mathrm{KO}}(\phi)$ of ϕ in the algebra

$$\mathrm{KO}^*_\Gamma(\tilde{T}^n)[e_{\mathrm{KO}}(\mathbb{H})^{-1}].$$

Remark

Since $\phi|_{V_0}$ defines the map from V_0 to W_0, we can "destabilizing" $\deg_{\mathrm{KO}}(\phi)$ with respect to $e_{\mathrm{KO}}(\tilde{\mathbb{R}}^2)$. Thus we do not need to add the inverse of $e_{\mathrm{KO}}(\tilde{\mathbb{R}}^2)$ to $\mathrm{KO}^*_\Gamma(\tilde{T}^n)[e_{\mathrm{KO}}(\mathbb{H})^{-1}]$.

Theorem

Suppose that ℓ is even. Under the isomorphism

$$\mathrm{KO}_{\Gamma}^{\ell-4k}(\tilde{T}^n)[e_{\mathrm{KO}}(\mathbb{H})^{-1}] \cong \bigoplus_{S\subset[n]} \mathrm{KO}_{\Gamma}^{\ell-4k}(\tilde{\mathbb{R}}^S)[e_{\mathrm{KO}}(\mathbb{H})^{-1}],$$

the KO-degree of $\phi : V \to W$ can be written as

$$[\deg_{\mathrm{KO}}(\phi)] = \sum_{S\subset[n]} \Lambda_S[\beta_S],$$

where $\beta_S \in \mathrm{KO}_{\Gamma}^{\ell-4k}(\tilde{\mathbb{R}}^S)$ and $\Lambda_S \in \mathbb{Z}$ are given by

$$\Lambda_S = \begin{cases} \pm N(S)2^{\lceil d_S/2\rceil+k-2} & d_S \equiv 1, 2 \bmod 4 \\ \pm N(S)2^{\lceil d_S/2\rceil+k-1} & d_S \equiv 3, 4 \bmod 4. \end{cases}$$

Borsuk-Ulam type inequality follows from the following lemma.

Lemma

Suppose that l is a positive **even** *integer. For each $S \subset [n]$, we have*

$$\ell_S \geq 2k_S + \varepsilon_0(d_S)$$

where $d_S = l - 4k - |S|$ and $\varepsilon_0(d)$ is defined by the following table and $\varepsilon_0(d+8) = \varepsilon_0(d)$.

d mod 8	0	1	2	3	4	5	6	7
$\varepsilon_0(d)$	2	3	4	3	4	3	2	1

Lemma

Lemma

Let E_1, E_2, F_1, F_2 be G-spin vector bundles. For any fibre-preserving proper G-map

$$\phi : E_1 \oplus E_2 \to F_1 \oplus F_2$$

satisfying $\phi(E_1) \subset F_1$, we have

$$\deg_{\mathrm{KO}}(\phi)\, e_{\mathrm{KO}}(E_2) = e_{\mathrm{KO}}(F_2)\, \deg_{\mathrm{KO}}(\phi|_{E_1}).$$

$$\begin{array}{ccc} E_1 \oplus E_2 & \xrightarrow{\phi} & F_1 \oplus F_2 \\ {\scriptstyle i_{E_1}} \uparrow & & \uparrow {\scriptstyle i_{F_1}} \\ E_1 & \xrightarrow[\phi|_{E_1}]{} & W_1 \end{array}$$

Proof of Lemma

By the definition of the KO-degree, we have

$$\deg_{\mathrm{KO}}(\phi)\, \mathrm{t}_{\mathrm{KO}}(E_1 \oplus E_2) = \phi^*(\mathrm{t}_{\mathrm{KO}}(F_1 \oplus F_2)).$$

Then

$$\begin{aligned} i_{E_1}^*(\text{L.H.S}) &= \deg_{\mathrm{KO}}(\phi) \deg_{\mathrm{KO}}(i_{E_1})\, \mathrm{t}_{\mathrm{KO}}(E_1) \\ &= \deg_{\mathrm{KO}}(\phi)\, \mathrm{e}_{\mathsf{KO}}(E_2)\, \mathrm{t}_{\mathrm{KO}}(E_1). \end{aligned}$$

On the other hand,

$$\begin{aligned} i_{E_1}^*(\text{R.H.S}) &= \phi^* i_{F_1}^*(\mathrm{t}_{\mathrm{KO}}(F_1 \oplus F_2)) \\ &= \phi^*(\deg_{\mathrm{KO}}(i_{F_1})\, \mathrm{t}_{\mathrm{KO}}(E_1)) \\ &= \mathrm{e}_{\mathsf{KO}}(F_2) \deg_{\mathrm{KO}}(\phi|_{E_1})\, \mathrm{t}_{\mathrm{KO}}(E_1) \end{aligned}$$

Comparing the coefficient of $\mathrm{t}_{\mathrm{KO}}(E_1)$, we obtain the required equation. □

Sketch of the proof

Step 1 Find Pin(2)-vector bundles representing $[V_1]$ and $[W_1]$ in terms of the Ksp-invariants $\{a_S\}_{S\subset[n]}$.

Step 2 Substitute the result of Step 1 for the equation

$$\deg_{\mathrm{KO}}(\phi)\, \mathsf{e}_{\mathsf{KO}}(V_1) = \mathsf{e}_{\mathsf{KO}}(W_1) \deg_{\mathrm{KO}}(\phi|_{V_1}).$$

Step 3 Calculate the KO-degree using the complexification map $\mathsf{KO}^*_\Gamma[\mathsf{e}_{\mathsf{KO}}(\mathbb{H})^{-1}] \to \mathsf{K}^*_\Gamma[\mathsf{e}_{\mathsf{KO}}(\mathbb{H})^{-1}]$.

Step 4 We obtain that for each S the coefficient Λ_S of β_S in the equation

$$[\deg_{\mathrm{KO}}(\phi)] = \sum_{S\subset[n]} \Lambda_S[\beta_S]$$

is an integer or an even integer. These conditions imply the required inequality.